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Hopf stars, twisted Hopf stars and scalar products on quantum spaces

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Abstract

The properties of Hopf star operations and twisted Hopf star operations on quantum groups are discussed in relation with the theory of representations (star representations). Invariant Hermitian sesquilinear forms (scalar products) on modules or module-algebras are then defined and analyzed. Particular attention is paid to scalar products that can be associated with the Killing form (when it exists) or with the left (or right) invariant integrals on the quantum group.

Our results are systematically illustrated in the case of a family of non-semisimple and finite dimensional quantum groups that are obtained as Hopf quotients of the quantum enveloping algebra $U_q(sl(2,\mathbb{C}))$, q being an Nth root of unity. Many explicit results concerning the case $N = 3$ are given.

We also mention several physical motivations for the present work: conformal field theory, spin chains, integrable models, generalized Yang–Mills theory with quantum group action and the search for finite quantum groups symmetries in particle physics.

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1. Introduction

The purpose of the present paper is to study the concepts of Hopf star operations and twisted Hopf star operations in the theory of quantum groups. This study is motivated by a number of physical considerations that we shall discuss in this section.

First of all, one should remember that the notion of quantum group (Hopf algebra) does not make use of a star operation — roughly speaking, the notion of complex conjugate — choosing comes only at a later stage. Such an operation is an antimultiplicative and antilinear involution which could be quite arbitrary when the (associative) algebra under consideration is not a Hopf algebra.

However, the existence of a coproduct allows to distinguish two particular kinds of star operations. The problem is to relate the star operations that one can define on the algebra H and on its tensor square $H \otimes H$, since we have a very special embedding of the first algebra into the latter one given by the coproduct. If $\Delta a = a_1 \otimes a_2$, it may be that the chosen star such that $\Delta(a^*) = a_1^* \otimes a_2^*$ (a Hopf star operation) but it also could happen that $\Delta(a^*) = a_2^* \otimes a_1^*$ (a twisted Hopf star operation).

Actually, one could define also "partially twisted stars", which in a sense continuously interpolates between a Hopf and a twisted star (see [1]), but these involve additional data, an element $f \in H \otimes H$.

In the case of Lie groups or Lie algebras, star operations are used to define real forms. However, for Hopf algebras the notion of "real form" is slightly more subtle (we shall say more about it later), but it is a priori clear that the notions of complex conjugate and of star representations should be discussed as soon as one wants to endow a representation space with some sort of scalar product.

A general discussion of star versus twisted star operations seems to be lacking in the literature: mathematical books on quantum groups (for instance [2] or [3]) only discuss (genuine) Hopf star operations, the same being true for all research papers studying C^*-algebra aspects of "matrix quantum groups" (in the sense of Woronowicz [4]). In the physics literature, most papers dealing with applications of quantum groups to integrable models, spin chains, or conformal field theory, usually do not choose any particular star operation at all on the quantum group of interest. But sometimes they do, and it turns out that the chosen star is often a twisted star — although usually the authors do not acknowledge the fact that it is so, [2] and this state of affairs creates some confusion. Quantum groups have also been discussed in relation with the possibility of q-deforming the Lorenz group, and here again, the two possibilities (twisted versus non-twisted) appear in the physical literature: from one side we have the papers [7] or [8], whereas from the other we have the papers [9,10].

Another motivation for our work comes from the possibility, as advocated by Connes [11], that reduced quantum groups like $U_q(sl(2,\mathbb{C}))$ at a cubic root of unity could have some essential role to play in the formulation of fundamental interactions (Standard Model). This suggestion is based upon the following two observations: first of all, when q is chosen to be a cubic root of unity the algebra of "functions" $Fun(SL_q(2,\mathbb{C}))$ is a Hopf–Galois

[2] With the notable exception of papers by Mack and Schomerus [5,6].

extension of $Fun(SL(2,\mathbb{C}))$ — the algebra of complex valued functions on the Lorenz group — the fiber being a finite dimensional quantum group $\mathcal{F}$ whose Hopf dual is a finite dimensional Hopf algebra quotient $U_q^{\mathrm{res}}(sl(2,\mathbb{C}))$ (that we shall call $\mathcal{H}$) of the quantum enveloping algebra $U_q(sl(2,\mathbb{C}))$. Next, for this q the semisimple part of $\mathcal{H}$ turns out to be isomorphic with the algebra $M(3,\mathbb{C})\oplus M(2,\mathbb{C})\oplus\mathbb{C}$. It is then tempting to use the tools of non-commutative geometry to build a physical model that would recover the usual Standard Model — may be a generalization of it — incorporating some action of an hitherto unnoticed finite quantum group of symmetries. The existence of a non-trivial coproduct mixing the different components of $\mathcal{H}$ and the nature of the representations of this non-semisimple Hopf algebra make it quite hard to recover the usual model of strong and electroweak interactions; this has not been achieved yet. In any case, it is clearly of interest to analyze in detail the structure of the representation theory of this Hopf algebra, and to pay particular attention to the different kinds of "reality" structures that one can find for these representations. For these reasons, and although we decided to write quite a general paper, most explicitly discussed examples will involve the case of the finite dimensional algebra $\mathcal{H} = U_q^{\mathrm{res}}(sl(2,\mathbb{C}))$ at a cubic root of unity.

Another motivation for studying the reality structures and the type of scalar products in star representations of quantum groups comes from our previous work [12,13]. Here, a new kind of gauge fields was obtained: starting from the observation that the reduced quantum plane (identified with the algebra of $N\times N$ complex matrices) is a module-algebra for the finite dimensional quantum group $\mathcal{H}$, when $q^N = 1$, we build a differential algebra over it by taking an appropriate quotient of the Wess–Zumino differential algebra over the — infinite dimensional — quantum plane; generalized differential forms are then obtained by making the tensor product of the De Rham complex of forms over an arbitrary space–time manifold times the previous Wess–Zumino reduced differential complex; generalized gauge fields (and curvatures, etc.) are finally constructed by standard non-commutative geometrical techniques. Clearly, we wish to construct a Lagrangian model involving the representations of a quantum group (that knows how to act on such generalized gauge fields), which requires the study of star (or twisted star) operations on the corresponding modules.

Finally, the last motivation comes from spin chains, integrable models and conformal theories. The q-parameter appearing in many conformal field theory models and integrable models is a primitive root of unity. Such values as a rule *exclude* the choice of a Hopf star operation leading to a compact quantum group like $SU_q(2)$, for instance. For this reason star operations used in papers like [14] — where the role of quantum groups is discussed in the context of spin chains, like $SU_q(2)$ in the *XXZ* model — are not true Hopf star operations; we shall return to this discussion in Section 5.

The structure of our paper is the following.

In Section 2, we gather information on stars operations: Hopf and twisted Hopf stars, compatible stars on modules and module-algebras, behavior under tensor product of representations, etc.

In Section 3, we discuss scalar products in representation spaces, its quantum invariance and associated star representations. As everywhere else in this paper, we first discuss all the general notions and then exemplify by taking the finite dimensional quantum group

$\mathcal{H} = U_q^{\text{res}}(sl(2,\mathbb{C}))$ for q a primitive odd root of unity (most of the time we take $N = 3$). The characteristics of the invariant scalar products on the irreducible and the projective indecomposable representations of $\mathcal{H}$ are studied in detail, both in the case where a genuine or a twisted Hopf star is chosen on $\mathcal{H}$. The same analysis is carried out for the module-algebra $M(N,\mathbb{C})$.

In Section 4, we examine more particularly the (left) regular representation of a Hopf algebra H and exhibit two distinguished invariant scalar products. The first one is defined in terms of the Killing form. The other is built using the left (or right) invariant integral on the algebra H. We then analyze in detail these scalar products for the case of $\mathcal{H}$. As we shall see, it happens that for many properties Hopf stars behave usually much better than twisted Hopf stars.

Appendix A summarizes what is needed for this paper from the structure and representation theory of the finite dimensional Hopf algebras $\mathcal{H} = U_q^{\text{res}}(sl(2,\mathbb{C}))$ when q is an odd primitive root of unity, in particular the structure of the projective indecomposable modules (PIMs) and of the corresponding irreducibles.

Appendix B recalls a few properties concerning the adjoint representation of quantum groups, together with the notions of quantum trace and quantum Killing form.

Appendix C gives a few explicit results concerning a “double cover” of the finite dimensional Hopf algebra $\mathcal{H}$.

1.1. About notations

F will generically denote a complex Hopf algebra, for example, the algebra of “functions” on a quantum group. H will be its dual (also a Hopf algebra), so that it can be thought of as a non-commutative generalization of the group-algebra of a finite group or as the non-commutative analog of the enveloping algebra of a Lie algebra. As already mentioned, the particular examples where H is chosen to be one of the finite dimensional quotients of $U_q(sl(2,\mathbb{C}))$ will be called $\mathcal{H}$. V will denote a representation space for H (and we shall have to specify if it is a left or a right module), and will therefore also be a (left or right) corepresentation space of the Hopf algebra F. Finally, M will denote a module-algebra for H (i.e., a comodule-algebra for F).

2. Stars

2.1. Hopf stars

Remember that a star on an algebra is an involutive antilinear antiautomorphism, i.e.,

$$(x^*)^* = x, \qquad (\lambda x)^* = \bar{\lambda} x^*, \qquad (xy)^* = y^* x^*, \quad \lambda \in \mathbb{C}.$$

Now, let the algebra on which $*$ acts be a complex Hopf algebra $H(m,\Delta,\eta,\epsilon,S)$. In this case one requires the star to satisfy two extra compatibility conditions [2] with the Hopf operations:

$$\Delta * = *_{\otimes} \Delta, \qquad \epsilon * = *_{\mathbb{C}} \epsilon. \tag{1}$$

However, the $*$'s on the right-hand side are operators on different spaces and are yet to be defined. $*_{\mathbb{C}}$ should be a star on $\mathbb{C}$, and therefore is just complex conjugation. The operation $*_\otimes$ should be an involution on $H \otimes H$, the standard choice is

$$*_\otimes = * \otimes *.$$

A star satisfying (1) with the standard choice of $*_\otimes$ is called a Hopf star, and in such a case H is called a Hopf star algebra.

Actually one could also make the choice $*_\otimes = \tau(* \otimes *)$, where τ is the tensorial flip (twisting); however, making such a choice and imposing (1) amounts to make the standard choice for $*_\otimes$ and rewrite (1) as

$$\Delta * = *_\otimes \Delta^{\mathrm{op}},$$

where $\Delta^{\mathrm{op}} \doteq \tau \circ \Delta$ is the opposite coproduct. We will call this second type of operation a *twisted* Hopf star, or even a twisted star. In this paper, therefore, we shall always make the standard choice for $*_\otimes$. In this section, we will analyze Hopf star algebras, leaving the study of the twisted case to Section 2.2.

Remark that there is no need to impose a relation between the star and the antipode (which is a *linear* antiautomorphism) because this one arises automatically. In fact, it is easy to see that

$$S * S* = id. \tag{2}$$

This is so because $* S^{-1} * = (* S *)^{-1}$ satisfies all the conditions for the antipode, which is unique. We should therefore remember that for Hopf star algebras

$$S* = * S^{-1}.$$

Notice that in general S has no reason to be equal to S^{-1} (imposing such a property would exclude all the Drinfeld–Jimbo deformations!).

Given a Hopf algebra H, one can consider its dual [3] Hopf algebra $F = H^\star$ with operations such that

$$\begin{aligned} &\langle \Delta f, h \otimes h' \rangle = \langle f, hh' \rangle, \qquad \forall h, h' \in H, \qquad \langle ff', h \rangle = \langle f \otimes f, \Delta h \rangle, \\ &\langle Sf, h \rangle = \langle f, Sh \rangle, \qquad \epsilon(f) = \langle f, \mathbb{1}_H \rangle, \qquad \langle \mathbb{1}_F, h \rangle = \epsilon(h), \end{aligned} \tag{3}$$

where $\langle , \rangle : F \otimes H \to \mathbb{C}$ is the bilinear evaluation pairing. When H is a Hopf star algebra, one may also define a dual star on F. By dual star we mean a star on F which is also a Hopf star. It is easy to verify that the following formula defines such an operation:

$$\langle f^*, h \rangle = \overline{\langle f, (Sh)^* \rangle} \quad \forall h \in H. \tag{4}$$

In what follows F will be thought of as the space of functions on a quantum group, and its dual H as the quantum group analog of the corresponding group algebra (or the enveloping algebra).

[3] The examples that we shall consider in this paper are finite dimensional (and non-semisimple) Hopf algebras, therefore it will be possible to identify canonically a given Hopf algebra with its bidual.

Remark that another standard accepted terminology for denoting the star structure of a (untwisted) Hopf star algebra is "real form on a Hopf algebra". This name does not imply, and we do not construct here, any *real* Hopf subalgebra of H, in the sense of being an algebra over the field $\mathbb{R}$ of real numbers (see [1] for a discussion of this point). Let T be a linear involutive Hopf algebra antiautomorphism (we call it T for transposition like in [15]) of a complex Hopf star algebra H, and consider the subspace $H_{\mathbb{R}} \doteq \{h \in H / h^* = T(h)\}$. Suppose moreover that $H = H_{\mathbb{R}} \oplus \mathrm{i}H_{\mathbb{R}}$, $T* = *T$ and $H_{\mathbb{R}}$ is invariant by the coproduct Δ (i.e., $\Delta H_{\mathbb{R}} \subset H_{\mathbb{R}} \otimes H_{\mathbb{R}}$), then $H_{\mathbb{R}}$ is a *real* Hopf algebra associated with the star $*$ and the involution T. Notice that $c \doteq T*$ is an antilinear involutive automorphism [4] and that $H_{\mathbb{R}}$ is the set of elements of H that are invariant under the conjugation c. Notice also that if $h \in H_{\mathbb{R}}$, then $\mathrm{i}h$, as defined in H, cannot belong to $H_{\mathbb{R}}$ since $(\mathrm{i}h)^* = -T(\mathrm{i}h)$. When H is "classical" (the enveloping algebra of some complex Lie algebra), such a $H_{\mathbb{R}}$ is the enveloping algebra of a real Lie algebra. Moreover, in this case one takes $T = S$ (since $S^2 = id$), so in $H_{\mathbb{R}}$ we have $x^* = T(x) = S(x) = x^{-1}$ for group-like elements and $x^* = T(x) = S(x) = -x$ for primitive elements.

2.1.1. Self-conjugate representations and compatible stars on modules

Suppose now that we are given a star $*_H$ on the Hopf algebra H, and a representation on a vector space V. We may have to face possible situations.

The first possibility is that we may want to define a star $*_V$ on V and decide to constrain it by imposing some sort of compatibility with the star $*_H$ on the quantum group. The second possibility is to suppose that we already start with a star $*_V$ on V (a priori given); in such a case [5] one can define on the same vector space a new representation called the conjugate representation. It may happen that both actions — the original one and its conjugate — are equivalent. In this last case the representation is therefore called *self-conjugated*.

Actually, the compatibility condition (see below) between the stars in the first scenario is just a particular case of the second option, as we define $*_V$ to be such that the representation precisely coincides with its conjugate.

Going back to our first problem, suppose now that we want to define a star $*_V$ on V, which is a representation space for the quantum group H and a corepresentation space for its dual F (i.e., V is a right F-comodule). Call the coaction $\delta_{\mathrm{R}} : V \mapsto V \otimes F$.

For a Hopf star $*_F$ on F it can be checked that the operation $\delta' \doteq (* \otimes *)\delta_{\mathrm{R}} * : V \to V \otimes F$ is again a right coaction on V. Therefore, it is natural to impose $\delta' = \delta_{\mathrm{R}}$ as the compatibility condition between the stars $*_F$ and $*_V$. With a slight abuse of notation we can even write

$$\delta_{\mathrm{R}}(z^*) = (\delta_{\mathrm{R}} z)^*, \quad z \in V, \tag{5}$$

where the conjugation on the right-hand side is the natural star structure on $V \otimes F$. In this case we may say that the star is covariant.

[4] In the case of our favorite example $\mathcal{H}$, such an operator c can be defined [16], on the generators, by setting $c(X_+) = -qKX_-$, $c(X_-) = -qK^{-1}X+$, $c(K) = K$.

[5] Thinking now only in the Hopf star case, as it is the only one where this notion makes sense.

V being a (right) F-comodule, it is also a (left) H-module. We have indeed an action $\triangleright : H \otimes V \mapsto V$ given by

$$h \triangleright z = (id \otimes \langle h, \cdot \rangle)\delta_{\mathrm{R}}(z).$$

Pairing Eq. (5) with an element $h \in H$, and using the duality of real structures we get the equation

$$h \triangleright z^* = [(Sh)^* \triangleright z]^*, \quad z \in V. \tag{6}$$

Assuming non-degeneracy of the duality pairing both expressions are completely equivalent, and imply some restrictions on $*_V$ given $*_F$ or $*_H$.

The action $h\triangleright$ of h on V is implemented due to an endomorphism $\rho[h]$ of this vector space, so one may also write $h \triangleright \doteq \rho[h]$. Using this notation, Eq. (6) can also be written as

$$\rho[h](z) = \bar{\rho}[h](z),$$

where $\bar{\rho}$ denotes the conjugate representation

$$\bar{\rho}[h](z) = [\rho[(Sh)^*](z^*)]^*$$

dual to the above δ' right coaction.[6] Therefore, the compatibility relation (6) between the stars on H and V can also be viewed as a very particular case of equivalency of representations: ρ and $\bar{\rho}$ should just coincide. Given the star operations, a representation ρ is called *self-conjugate* if there exists an invertible operator $U : V \mapsto V$ such that

$$U^{-1}\rho[h]U = \bar{\rho}[h].$$

Up to now we did not assume that the representation space V was endowed with a scalar product $(\cdot, \cdot)$. Therefore, we cannot impose, at this point, that U should be unitary. We cannot assume either that the star operation on V is an antiunitary operator, $(v^*, w^*) = (w, v)$. For the same reason too, the notation † (adjoint) was avoided. In any case, a Hopf algebra is, in particular, an associative algebra, and if it so happens that a *real* Hopf algebra $H_{\mathbb{R}}$ can be defined the usual classification for representations of real associative algebras on complex Hilbert spaces will, of course, also hold. We could have three types of representations, complex, real, and quaternionic; we refer the reader to standard textbooks (see for instance [17,18]).

2.1.2. Compatible stars on module-algebras

Instead of a comodule V, we now take a right F-comodule-algebra M, i.e., we assume that the right coaction δ_{R} is an algebra homomorphism from M to $M \otimes F$,

$$\delta_{\mathrm{R}}(zw) = \delta_{\mathrm{R}} z \delta_{\mathrm{R}} w.$$

The map $\delta' : M \to M \otimes F$ defined as above will again be an algebra homomorphism, i.e., $\delta'(zw) = \delta' z\, \delta' w$. Thus Eq. (5) is still a good requirement when the comodule M is an

[6] Remember that in the "classical" case (i.e., real forms of Lie algebras and their enveloping algebras), $(Sh)^* = h$ for the Lie algebra generators, and we recognize the usual equation $\bar{\rho} = *\rho*$ defining the conjugate representation.

algebra and shows that compatibility of the coaction with a given Hopf star operation needs only to be verified on the (algebra) generators. [7]

Obviously the dual equation (6) defining compatibility of Hopf stars on left modules will also have the same properties. Remember that, being a right F-comodule-algebra, M supports a left action of the dual H of F and indeed this action is compatible with the product in M (call $\Delta h = h_1 \otimes h_2$):

$$h \triangleright (zw) = (h_1 \triangleright z)(h_2 \triangleright w).$$

2.1.3. *Example of the reduced $SL_q(2,\mathbb{C})$ at $q^N = 1$*

Hopf stars on $\mathcal{F}$ and $\mathcal{H}$

First of all, remember that in the quantum case one has three possibilities for the star operations on $Fun(SL_q(2,\mathbb{C}))$ (up to star-Hopf homomorphisms). Given the conventions chosen in [13], they are given on generators by the following.

- The real form $Fun(SU_q(2))$: $a^* = d$, $b^* = -qc$, $c^* = -q^{-1}b$ and $d^* = a$. Moreover, q should be real.
- The real form $Fun(SU_q(1,1))$: $a^* = d$, $b^* = qc$, $c^* = q^{-1}b$ and $d^* = a$. Moreover, q should be real.
- The real form $Fun(SL_q(2,\mathbb{R}))$: the conjugation is given by

$$a^* = a, \qquad b^* = b, \qquad c^* = c, \qquad d^* = d. \tag{7}$$

Here, q can be complex but it should be a phase.

When $q = \pm\mathrm{i}$ — hence $q^4 = 1$ — there are still two other Hopf star structures that have no classical limit (see [2] and references therein). A systematic analysis of real forms for special linear quantum groups $SL_q(n)$ was made by Jain and Ogievetsky [19], and in the case of $GL_{p,q}(2)$ or $GL_\alpha^J(2)$ by Ewen et al. [20].

It is already clear from these results that taking q a root of unity is incompatible with the SU_q and $SU_q(1,1)$ real forms. The only possibility if we assume $q^N = 1$ is to choose the Hopf star corresponding to $Fun(SL_q(2,\mathbb{R}))$. Moreover, in such a case the star is compatible with the finite dimensional Hopf algebra quotient $\mathcal{F}$ obtained by factoring this quantum group by the Hopf ideal defined by [13]: $a^N = d^N = \mathbb{1}$, $b^N = c^N = 0$ (take N odd here, and q a primitive Nth root of unity).

The corresponding dual star on the dual Hopf algebra $U_q(sl(2,\mathbb{C}))$ (see [13] or Appendix A for its structure) is

$$X_+^* = -q^{-1}\, X_+, \qquad X_-^* = -q\, X_-, \qquad K^* = K. \tag{8}$$

Here, one can also factor the quantum enveloping algebra by the Hopf ideal defined by $K^N = \mathbb{1}$, $X_+^N = 0$, $X_-^N = 0$ and the same remarks concerning the fact that the stars passes to the quotient $\mathcal{H}$ apply [13].

Compatible star on the quantum plane $\mathcal{M}$

The quantum group $Fun(SL_q(2,\mathbb{C}))$ coacts on the quantum plane algebra generated by x, y such that $xy = qyx$. For a root of unity this algebra can be quotiented by the

[7] In the case of a module-algebra, the star operation is of course assumed to be antimultiplicative ($(xy)^* = y^*x^*$).

ideal defined by $x^N = y^N = \mathbb{1}$ to obtain a finite dimensional algebra that we call $\mathcal{M}$. $\mathcal{M}$ is a right comodule-algebra for $\mathcal{F}$ and the right coaction is given by $\delta_{\mathrm{R}}(x \quad y) = (x \quad y)\dot{\otimes}\begin{pmatrix} a & b \\ c & d \end{pmatrix}$. The reduced quantized universal enveloping algebra $\mathcal{H}$ acts on this quantum plane (for compatible formulae for actions and coactions, see for instance [13]). Up to equivalences (now $*$-homomorphisms) there is only one conjugation on this quantum plane compatible with the requirements (5) or (6). It works for both the infinite dimensional algebra or its reduced (finite) quotients when $q^N = 1$. It is

$$x^* = x, \qquad y^* = y.$$

Notice that although the star is the identity on the generators, it is non-trivial on $\mathcal{M}$ since it is an antimultiplicative operation and, for instance, $(xy)^* = q^{-1}xy$.

2.2. *Twisted Hopf stars*

As we mentioned before there is an alternative way of relating the Hopf and star structures on a Hopf algebra. It reduces to replacing in (1) the equation for the coproduct by [8]

$$\Delta * = (* \otimes *)\Delta^{\mathrm{op}}.$$

Given such a twisted star on a Hopf algebra H, the dual Hopf algebra $F = H^\star$ can be also endowed with a dual twisted Hopf star. One just has to define it by

$$\langle f^*, h\rangle = \overline{\langle f, h^*\rangle}. \tag{9}$$

It can be readily verified that this operation is a twisted Hopf star on F. As in the untwisted case, a relation involving the antipode is automatically fulfilled. Now the antipode and the star commute,

$$S* = *S. \tag{10}$$

This is so because $*S*$ is again an antipode, which is unique.

2.2.1. *Compatible twisted stars on modules*

Let V be again a right F-comodule. Given $*_F$ a twisted Hopf star on F we would now like to use it to restrict the possible choices for a star $*_V$ on V, as it was done with Eq. (5) in the pure Hopf case.

$*_F$ being twisted, $(* \otimes *)\, \delta_{\mathrm{R}} * : V \mapsto V \otimes F$ is no longer a right coaction, however, $\tau(*\otimes*)\, \delta_{\mathrm{R}} * : V \mapsto F \otimes V$ is a left one. Moreover, $(id\otimes S)(*\otimes*)\, \delta_{\mathrm{R}} * = (*\otimes*)(id\otimes S)\, \delta_{\mathrm{R}} *$ is again a right coaction. Consequently, we may require

$$(id \otimes S)\delta_{\mathrm{R}}(z^*) = (\delta_{\mathrm{R}} z)^*, \quad z \in V, \tag{11}$$

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[8] It could even be written as $\Delta * = *_{\mathrm{op}}\Delta$ at the expense of using a flipped definition of the star on the tensor product: $*_{\mathrm{op}}(f \otimes g) = g^* \otimes f^*$.

or the following dual expression for the corresponding action of H on the module V:

$$h \triangleright z^* = [(Sh)^* \triangleright z]^*, \qquad z \in V. \tag{12}$$

Notice that this condition looks formally like (6).

2.2.2. Compatible twisted stars on module-algebras

If we now let V to be an F-comodule-algebra (we then call it M rather than V), it happens that (11) is not a reasonable condition anymore, because $(id \otimes S)\delta_R *$ and $*\delta_R$ have different homomorphism behavior. It may also be said that $(id \otimes S)(* \otimes *)\delta_R *$ is not an R-coaction *on an algebra* but only a coaction; it does not preserve the product on M.

As $\tau(* \otimes *)\,\delta_R *$ is a good homomorphism, the way out to constrain $*_M$ is to choose some *other* left algebra-coaction δ_L on M and impose

$$\delta_R(z^*) = (\delta_L z)^{*_{op}}, \quad z \in M, \tag{13}$$

where now the star $*_{op}$ on the right-hand side includes the tensorial flip (on $F \otimes M$ it is given by $*_{op}(f \otimes z) = z^* \otimes f^*$, $z \in M$, $f \in F$). Remark that for many interesting cases we have both natural left and right coactions; this is for instance the case for quantum planes.

The dual condition involves the left and right actions of H on M which are dual to δ_R and δ_L, they are, respectively, denoted by $\triangleright$ and $\triangleleft$. It reads

$$z^* \triangleleft h = [h^* \triangleright z]^*, \quad h \in H, \ \ z \in M. \tag{14}$$

2.2.3. Example of the reduced $SL_q(2,\mathbb{C})$ at $q^N = 1$

Twisted Hopf stars on $\mathcal{F}$ and $\mathcal{H}$

On both the reduced and unreduced $SL_q(2,\mathbb{C})$, the twisted stars are essentially the following[9] (i.e., up to automorphisms):

$$a^* = a, \qquad b^* = \pm c, \qquad c^* = \pm b, \qquad d^* = d. \tag{15}$$

So we have two of them, and the corresponding dual twisted stars are given by

$$X_+^* = \pm X_-, \qquad X_-^* = \pm X_+, \qquad K^* = K^{-1}. \tag{16}$$

Thus we see that, when q is a root of unity, these twisted stars allow one to recover the $SU(2)$ (+ sign) and $SU(1,1)$ (− sign) real forms, something that would be otherwise forbidden with a *true* Hopf star operation.

Compatible star on the quantum plane $\mathcal{M}$

On the quantum plane there is, again up to equivalence, only one star structure compatible in the sense (13) or (14) with each of the twisted stars (15) or (16). These twisted stars are, respectively, given by

$$x^* = x, \qquad y^* = \pm y. \tag{17}$$

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[9] The operation defined on generators by $a^* = d$, $d^* = a$, $b^* = \pm b$ and $c^* = \pm c$ “almost works”, in the sense that it defines a twisted star in $GL_q(2,\mathbb{C})$ but it is incompatible with the determinant condition defining $SL_q(2,\mathbb{C})$.

2.3. *Stars and tensor products*

2.3.1. *Tensor product of matrices*

If

$$m = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \quad \text{and} \quad M = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$$

are two matrices with *non-commutative* entries belonging to a ring $\mathcal{B}$, then it is standard to define their tensor product as

$$m \otimes M \doteq \begin{pmatrix} aA & aB & bA & bB \\ aC & aD & bC & bD \\ cA & cB & dA & dB \\ cC & cD & dC & dD \end{pmatrix}.$$

We now define a different tensor product, $\otimes_{\mathrm{op}}$, by

$$M \otimes_{\mathrm{op}} m = \begin{pmatrix} Aa & Ba & Ab & Bb \\ Ca & Da & Cb & Db \\ Ac & Bc & Ad & Bd \\ Cc & Dc & Cd & Dd \end{pmatrix},$$

the difference being that now the matrix which determines the coarse structure of the tensor product is the second one. It is clear and well known that $m \otimes M \neq M \otimes m$, independent of whether $\mathcal{B}$ is commutative or not. However, we see that when $\mathcal{B}$ is abelian, $m \otimes M = M \otimes_{\mathrm{op}} m$. The previous calculation tells us how to modify this result when $\mathcal{B}$ is not commutative: calling $\mathcal{B}^{\mathrm{op}}$ the same ring with *opposite* multiplication (so that $A \cdot_{\mathrm{op}} a = a.A$, for example), we obtain

$$m(\mathcal{B}) \otimes M(\mathcal{B}) = M(\mathcal{B}^{\mathrm{op}}) \otimes_{\mathrm{op}} m(\mathcal{B}^{\mathrm{op}}),$$

where the notation $M(\mathcal{B}^{\mathrm{op}}) \otimes_{\mathrm{op}} m(\mathcal{B}^{\mathrm{op}})$ means that we first take the opposite tensor product of the two matrices and subsequently we multiply the matrix elements in the opposite order.

Suppose in addition that the ring $\mathcal{B}$ is endowed with a star operation $*$, and call $\dagger$ the conjugation of matrices with $\mathcal{B}$-entries. In the case of 2×2 matrices, this reads

$$\begin{pmatrix} a & b \\ c & d \end{pmatrix}^{\dagger} \doteq \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

So defined $\dagger$ is antimultiplicative. Moreover, direct calculation shows that

$$(m \otimes M)^{\dagger} = M^{\dagger} \otimes_{\mathrm{op}} m^{\dagger}.$$

When $\mathcal{B}$ is commutative, the previous right-hand side can be written simply as $m^{\dagger} \otimes M^{\dagger}$.

2.3.2. Tensor product of representations

Now let $\mathcal{A}$ be an algebra. Take ρ_1 and ρ_2 to be two representations of $\mathcal{A}$ in vector spaces V_1 and V_2. Then, once bases are chosen, $\rho_1(a)$ and $\rho_2(a)$, with $a \in \mathcal{A}$, are two matrices with commutative entries.

It is clear that $\rho_1 \otimes \rho_2$ is a representation of the algebra $\mathcal{A} \otimes \mathcal{A}$, indeed, with $a \otimes b \in \mathcal{A} \otimes \mathcal{A}$, we have

$$[\rho_1 \otimes \rho_2](a \otimes b) = \rho_1(a) \otimes \rho_2(b).$$

However, this is not a representation of $\mathcal{A}$, unless we have a coproduct (algebra homomorphism) from $\mathcal{A}$ to $\mathcal{A} \otimes \mathcal{A}$: using

$$a \in \mathcal{A} \to \Delta a \doteq a_1 \otimes a_2 \in \mathcal{A} \otimes \mathcal{A},$$

one defines $\rho_1 \otimes \rho_2$ as a representation of $\mathcal{A}$ by setting

$$[\rho_1 \otimes \rho_2][a] \doteq [\rho_1 \otimes \rho_2](\Delta a).$$

If $\mathcal{A}$ is a Hopf algebra, we are in such a situation. This is what we assume from now on.

Now, suppose that $\mathcal{A}$ has a star operation, and that $(\rho_1, \dagger)$, $(\rho_2, \dagger)$ are star representations of this Hopf algebra on modules V_1, V_2 (each one endowed with a scalar product for which the adjoint is denoted by $\dagger$). So, we have

$$\rho_1(u^*) = (\rho_1(u))^\dagger \quad \text{and} \quad \rho_2(u^*) = (\rho_2(u))^\dagger.$$

We shall now suppose that the star is, somehow, compatible with the Hopf structure. We shall discuss the Hopf star and twisted Hopf star cases.

We first suppose that $*$ is a Hopf star. It then commutes with Δ, and

$$\begin{aligned}[\rho_1 \otimes \rho_2][a^*] &= [\rho_1 \otimes \rho_2](\Delta a^*) = [\rho_1 \otimes \rho_2](*\Delta a) = [\rho_1 \otimes \rho_2](a_1^* \otimes a_2^*)\\ &= \rho_1(a_1^*) \otimes \rho_2(a_2^*) = (\rho_1(a_1))^\dagger \otimes (\rho_2(a_2))^\dagger = (\rho_1(a_1) \otimes \rho_2(a_2))^\dagger\\ &= ([\rho_1 \otimes \rho_2](a_1 \otimes a_2))^\dagger = ([\rho_1 \otimes \rho_2](\Delta a))^\dagger = ([\rho_1 \otimes \rho_2][a])^\dagger.\end{aligned}$$

Therefore, $\rho_1 \otimes \rho_2$ is also a $*$-representation.

We now suppose that $*$ is a twisted Hopf star. It no longer commutes with Δ but intertwines it with the opposite coproduct Δ^{op}. In this case,

$$\begin{aligned}[\rho_1 \otimes \rho_2][a^*] &= [\rho_1 \otimes \rho_2](\Delta a^*) = [\rho_1 \otimes \rho_2](*\Delta^{\mathrm{op}} a) = [\rho_1 \otimes \rho_2](a_2^* \otimes a_1^*)\\ &= \rho_1(a_2^*) \otimes \rho_2(a_1^*) = (\rho_1(a_2))^\dagger \otimes (\rho_2(a_1))^\dagger = (\rho_1(a_2) \otimes \rho_2(a_1))^\dagger\\ &= ([\rho_1 \otimes \rho_2](a_2 \otimes a_1))^\dagger = ([\rho_1 \otimes \rho_2](\Delta^{\mathrm{op}} a))^\dagger \neq ([\rho_1 \otimes \rho_2][a])^\dagger.\end{aligned}$$

Therefore, $\rho_1 \otimes \rho_2$ is not a $*$-representation for a twisted $*$. However, we have the possibility of defining "another" tensor product of representations, [10] called $\otimes_{\mathrm{op}}$, as follows:

$$[\rho_1 \otimes_{\mathrm{op}} \rho_2][a] \doteq [\rho_1 \otimes \rho_2](\Delta^{\mathrm{op}} a).$$

---

[10] When $\mathcal{A}$ is quasitriangular, we recall that the two coproducts are related by an R-matrix as follows: $\Delta^{\mathrm{op}}(a) = R\Delta(a)R^{-1}$.

With this notation at hand, we can write

$$[\rho_1 \otimes \rho_2][a^*] = ([\rho_1 \otimes_{\mathrm{op}} \rho_2][a])^\dagger.$$

For this reason, "true" Hopf stars are usually preferred in mathematics, as the category of $*$-representations is closed under tensor product. Another possibility, the one employed in CFTs, is to truncate tensor products (see Section 5). Star representations are closed under this truncated tensor product for both types of stars.

2.3.3. Hopf action on vectors with non-commutative elements

We now suppose that ρ_1 and ρ_2 are no longer complex matrices but matrices with elements taken in a star algebra $\mathcal{B}$. We still assume that we have a left action, in the sense $\rho_i(ab) = \rho_i(a)\rho_i(b)$, but this is not a representation in the usual sense. As before we assume that $\mathcal{A}$ is endowed with a star operation and that $(\rho_i, \dagger)$ are star representations, in the sense $\rho(a^*) = (\rho(a))^\dagger$, where $\dagger$ transposes the matrix $\rho(a)$ and takes the conjugate (in $\mathcal{B}$) of each element.

If we suppose that $*_{\mathcal{A}}$ is a Hopf star, then a direct calculation shows that

$$[\rho_1 \otimes \rho_2][a^*] = ([\rho_1^{\mathrm{op}} \otimes \rho_2^{\mathrm{op}}][a])^\dagger.$$

Usually, for $\mathcal{B} = \mathbb{C}$, we have $\rho^{\mathrm{op}} = \rho$, but this is not so in general; the upper index "op" in $\rho^{\mathrm{op}}(a)$ reminds us that we should use the opposite multiplication of $\mathcal{B}$ when making product of matrices such as $\rho^{\mathrm{op}}(a)$.

If we take instead a twisted Hopf star, the conclusion is now

$$[\rho_1 \otimes \rho_2][a^*] = ([\rho_1^{\mathrm{op}} \otimes_{\mathrm{op}} \rho_2^{\mathrm{op}}][a])^\dagger.$$

3. Invariant scalar products

3.1. Compatibility with Hopf stars

Defining the notion of an invariant scalar product $(\cdot, \cdot)$ on a representation space V of a quantum group H is not as straightforward as in the classical case. We want the scalar product to commute with the action of the Hopf algebra in the appropriate sense. However, in order to get a relation which needs to be checked only on the quantum group generators, we want this condition to be a (linear) homomorphism in the H variable. Given that the scalar product is antilinear in one of its variables, there are two ways of achieving this, [11]

$$\epsilon(h)(z, w) = ((* S h_1) \triangleright z, h_2 \triangleright w) \tag{18}$$

or

$$\epsilon(h)(z, w) = ((S * h_1) \triangleright z, h_2 \triangleright w). \tag{19}$$

We refer the reader to [12] for more detailed discussion.

[11] In the "classical case" (real form of some Lie algebra), both formulae read $(z, h \triangleright w) + (h \triangleright z, w) = 0$.

As the Hopf star does not commute with the antipode, since $S * = * S^{-1}$, (18) and (19) are, in general, two different conditions.

For the scalar product to be invariant in the sense of Eq. (18), one only needs the quantum group action to be given by a $*$-*representation*:[12]

$$(h \triangleright z, w) = (z, h^* \triangleright w). \tag{20}$$

Notice that (20) implies (18) but not conversely. In the same way the alternative requirement $(h \triangleright z, w) = (z, S^2(h^*) \triangleright w)$ implies that condition (19) is satisfied. However, in our examples, we will choose to work with $*$-representations, and therefore, with invariant scalar products in the sense (18).

Assuming a non-degenerate pairing between H and its dual F, and extending the notation $(\cdot, \cdot)$ to the following F-valued sesquilinear map on $V \otimes F$:

$$(v \otimes f, w \otimes g) \doteq (v, w) f^* g, \quad v, w \in V, \quad f, g \in F,$$

we may write the previous equations in the dual picture in a very simple way. The first invariance condition reads

$$(\delta_{\mathrm{R}}\, v, \delta_{\mathrm{R}}\, w) = (v, w) \mathbb{1}_F,$$

whereas the $*$-representation requirement (20) reads

$$(v, \delta_{\mathrm{R}} w) = ((id \otimes S)\delta_{\mathrm{R}} v, w).$$

Again, this latter requirement implies the former.

Now, let $\{v_i\}$ be a basis of the vector space V, and call $G_{ij} = (v_i, v_j)$ the corresponding metric. Moreover, define the matrix of $h \in H$ in such a basis by $h \triangleright v_i \doteq \|h\|_{ji} v_j$. From Eq. (20) it is now trivial to get the matrix identities

$$\|h\|^{\dagger} G = G \|h^*\|, \tag{21}$$

where $\dagger$ denotes the transposed conjugate matrix. In particular, for an orthonormal basis this reduces to $\|h\|^{\dagger} = \|h^*\|$.

3.2. *Compatibility with twisted Hopf stars*

The previous discussion (Section 3.1) does not use the fact that the chosen star should be a “true” Hopf star operation; therefore, the same invariance conditions (18) and (19) still apply in the twisted case. However, now $S* = *S$, so that both conditions coincide.

The invariance requirement is still automatically satisfied if the representation of H under study is a $*$-representation (formula (20)). However, now the dual formulas are slightly different due to the absence of the antipode in the duality (9). The scalar product will be called invariant if

$$((id \otimes S)\delta_{\mathrm{R}}\, v, \delta_{\mathrm{R}}\, w) = (v, w) \mathbb{1}_F,$$

[12] If the action of h is implemented by a linear operator $\rho[h]$ on V, this condition simply reads $\rho[h^*] = (\rho[h])^{\dagger}$, where $\dagger$ is the usual adjoint operator.

and the (co)representation will be a $*$-(co)representation if

$$(v, \delta_{\mathrm{R}} w) = (\delta_{\mathrm{R}} v, w).$$

Selecting a basis of V we can write, exactly as in the untwisted case, $\|h\|^{\dagger} G = G\|h^*\|$ for any $h \in H$.

3.3. *Quantum metric and quantum symplectic form on* $M(2, \mathcal{F})$

Untwisted case

The q-deformed symplectic form in two dimensions (one may call it the q-deformed epsilon tensor) is given by the matrix

$$\Sigma \doteq \begin{pmatrix} 0 & q^{-1/2} \\ -q^{1/2} & 0 \end{pmatrix}.$$

In fact the $*$-representation condition implies for the true Hopf star case the equation

$$T^{\dagger}\, \Sigma\, T = \Sigma.$$

Here,

$$T \doteq \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

is the multiplicative matrix of generators of the quantum function group $SL_q(2, \mathbb{C})$, and the $\dagger$ operation corresponds to applying $*$ to the elements and transposing the matrix:

$$T^{\dagger} \doteq \begin{pmatrix} a^* & c^* \\ b^* & d^* \end{pmatrix}.$$

Notice that the above equation is different from (21) as there is a duality involved, there $h \in H$, $\|h\|_{ij} \in \mathbb{C}$, whereas here $T_{ij} \in F$. Using the star (7) corresponding to $SL_q(2, \mathbb{R})$ and fixing a global factor by requiring hermiticity of Σ, we finally obtain the "invariant metric" given above.

Twisted case

Now, as the duality between the star on a Hopf algebra and its dual differs from the one in the untwisted case, the $*$-representation condition implies the relation

$$(ST)^{\dagger}\, \Sigma\, T = \Sigma,$$

where S is the antipode. Taking the twisted conjugacy $a^* = a$, $b^* = \pm c$, $c^* = \pm b$ and $d^* = d$, we get the metric

$$\Sigma_{\pm} \doteq \begin{pmatrix} 1 & 0 \\ 0 & \pm 1 \end{pmatrix},$$

as we would expect in a (twisted) $SU(2)$ and $SU(1, 1)$ case, respectively.

3.4. Invariant scalar products for $\mathcal{H}$ endowed with a Hopf star

3.4.1. Invariant scalar products on the indecomposable representations of $\mathcal{H}$

This was worked out in Appendix E of [13]. Here, we repeat the expressions for the matrices of scalar products G just for completeness and to ease the comparison with the twisted case. Technically this is done by solving the set of linear equations (21) for the coefficients G_{ij} taking $h = X_\pm$ and K, and imposing hermiticity of G. Each entry in the list below corresponds to an indecomposable representation, remember that 3_{irr} is projective and irreducible, whereas 6_{odd} and 6_{eve} are projective indecomposable (with corresponding irreducible representations of dimensions 1 and 2, respectively). We only single out the following salient features (notice that G is always given up to an overall normalization factor):

- 3_{irr}: we get

$$G = \begin{pmatrix} 0 & 0 & -q^2 \\ 0 & 1 & 0 \\ -q & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = (++-).$$

  The index of G (maximal dimension of each of the two maximally isotropic subspaces) is therefore 1, and the Witt decomposition reads $3 = 1 + 1 + 1$.

- 6_{odd}: with $\beta \in \mathbb{R}$ we have here

$$G = \begin{pmatrix} 0 & 0 & 0 & q & 0 & 0 \\ 0 & 0 & -q & 0 & 0 & 0 \\ 0 & -q^2 & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix} \sim \mathit{Diag}(1, 1, -1, -1, \lambda_+, \lambda_-),$$

  with $\lambda_+ > 0$, $\lambda_- < 0$. G is neutral, as its signature is $\sigma = (+++---)$. The index of G is 3 and the Witt decomposition reads $6 = 3 + 3$.

- 5_{odd}: taking $\beta, \gamma \in \mathbb{R}$, $g \in \mathbb{C}$,

$$G = \begin{pmatrix} 0 & 0 & \mathrm{i}q\gamma & g & 0 \\ 0 & 0 & -q^2\bar{g} & \mathrm{i}q\beta & 0 \\ -\mathrm{i}q^2\gamma & -qg & 0 & 0 & 0 \\ \bar{g} & -\mathrm{i}q^2\beta & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix} \sim \mathit{Diag}(\lambda_+, -\lambda_+, \lambda_-, -\lambda_-, 0),$$

  and $\sigma = (++--0)$.

- 3_{odd}: now,

$$G = \begin{pmatrix} 0 & \mathrm{i}q & 0 \\ -\mathrm{i}q^2 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} \quad \text{and} \quad \sigma = (+-0).$$

- 6_{eve}: the metric should be ($\beta \in \mathbb{R}$)

$$G = \begin{pmatrix} 0 & 0 & \mathrm{i}q\beta & -\mathrm{i}q & 0 & 0 \\ 0 & 0 & -\mathrm{i}q & 0 & 0 & 0 \\ -\mathrm{i}q^2\beta & \mathrm{i}q^2 & 0 & 0 & 0 & 0 \\ \mathrm{i}q^2 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & \mathrm{i} \\ 0 & 0 & 0 & 0 & -\mathrm{i} & 0 \end{pmatrix},$$

  with a signature $\sigma = (+++---)$ for any β. As in the 6_{odd} case, G is neutral with an index of 3, and the Witt decomposition reads $6 = 3 + 3$.
- 4_{eve}: having $\alpha, \beta \in \mathbb{R}$, and $g \in \mathbb{C}$, we may write

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & g \\ 0 & 0 & \bar{g} & \beta \end{pmatrix}.$$

  Now, its signature obviously depends on the parameters.
- 3_{eve}: here we have simply $G = Diag(0, 0, 1)$, and $\sigma = (+00)$.
- 2_{eve}: in this case,

$$G = \begin{pmatrix} 0 & \mathrm{i}q \\ -\mathrm{i}q^2 & 0 \end{pmatrix} \sim Diag(1, -1).$$

3.4.2. Invariant scalar products on $\mathcal{M}$

It can be seen that for $N = 3$, $q^3 = 1$, the reduced quantum plane $\mathcal{M}$, a module-algebra for $\mathcal{H}$, is isomorphic as an algebra to the matrix algebra $M(3, \mathbb{C})$, whereas as a vector space splits into the sum of three *unequivalent* indecomposable representations, namely $\mathcal{M} \sim 3_{\text{irr}} \oplus 3_{\text{eve}} \oplus 3_{\text{odd}}$.

Actually, this feature can be generalized for all N odd, $q^N = 1$. The corresponding quantum plane (which is now isomorphic with $M(N, \mathbb{C})$) splits into the sum of N *unequivalent* indecomposable representations of $\mathcal{H}$. One of them is the irreducible N_{irr}, and the others are analogous to the "intermediate modules" that appear within each lattice of submodules associated to the other $N - 1$ PIMs of $\mathcal{H}$. This property was proven in [21].

A word of warning seems to be necessary here: the algebra $M(N, \mathbb{C})$ plays an ubiquitous role here. Indeed, on one hand it is isomorphic with a simple subalgebra of $\mathcal{H}$ (see the structure of the regular representation given in Appendix A). As such, its underlying vector space splits into a sum of N subspaces carrying equivalent representations (all equivalent to the N_{irr}), appearing in the decomposition of the regular representation in PIMs. In this way $M(N, \mathbb{C})$ appears as an algebra and as a module, but not as a module-algebra (considering $Z, W \in M(N, \mathbb{C}) \subset \mathcal{H}$ and $X \in \mathcal{H}$, in general $X(ZW) \neq (X_1 Z)(X_2 W)$). On the other hand, $M(N, \mathbb{C})$ is also isomorphic with the reduced quantum plane, and as such it is a module-algebra, but not a subalgebra of $\mathcal{H}$ anymore. Its decomposition under the action of $\mathcal{H}$ is now more subtle, since it reads $M(N, \mathbb{C}) \sim N_{\text{irr}} \oplus N_1 \oplus N_2 \oplus \cdots \oplus N_{N-1}$.

Because of this last result, one could be tempted to think that the most general scalar product on the reduced quantum plane $\mathcal{M}$ is simply given by the direct sum of its restrictions to the modules 3_{irr}, 3_{eve} and 3_{odd} that are already known, but it is not so. Indeed, non-diagonal blocks may appear as we can have non-zero projections amongst vectors of different indecomposable representations.

$\mathcal{M}$ being not only a module but also a module algebra, we impose condition (20) for the left action of $\mathcal{M}$ on itself given by multiplication as well. This singles out a unique invariant Hermitian form $(\ ,\ )$ up to an overall scaling factor. Its structure was studied in Section 5.5 of [13] and goes as follows: the only non-zero scalar products are those of the type $(x^r y^s, x^p y^t)$ with $r + p = s + t = 2$, and they are all determined by setting $(xy, xy) = 1$. The signature of this metric is $(5+, 4-)$, so its index is 4 and the Witt decomposition reads $9 = 4+4+1$. In the basis $\{\{x^2, xy, y^2\}, \{x, y, x^2y^2\}, \{\mathbb{1}, x^2y, xy^2\}\}$, the scalar product can be written as

$$G = \begin{pmatrix} B & 0 & 0 \\ 0 & 0 & B \\ 0 & B & 0 \end{pmatrix},$$

where B is the 3×3 block:

$$B = \begin{pmatrix} 0 & 0 & q^2 \\ 0 & 1 & 0 \\ q & 0 & 0 \end{pmatrix}.$$

The restriction of this scalar product to the subspace 3_{irr} coincides with what was already obtained before, a form of signature $(2+, 1-)$. The restriction to the subspaces 3_{eve} and 3_{odd} is actually totally degenerate, so the conclusion we find for $\mathcal{M}$ does not contradict what was already obtained for 3_{eve} and 3_{odd} (just choose an overall scaling factor equal to 0 in the latter cases).

3.4.3. Invariant scalar products on the regular representation of $\mathcal{H}$

One should not be tempted to think that the most general Hermitian scalar product on $\mathcal{H}(N = 3)$ itself is simply given by its restrictions to the direct sum $3[3_{\text{irr}}]\oplus 2[6_{\text{eve}}]\oplus 1[6_{\text{odd}}]$ since we may very well accept "off-block" components. As a matter of fact, the constraints in this case are rather weak: for any given star, any Hermitian form such that $(X^*Y, Z) = (Y, XZ)$ will work, but such a form is totally determined by the values of $(\mathbb{1}, X_+^a X_-^b K^c)$. Since we have N^3 terms, we see at once that the most general invariant scalar product on $\mathcal{H}$ will depend on N^3 parameters (real ones, due to the hermiticity of the scalar product). If one really wants to obtain an explicit expression for the possible metrics G's, in the case $N = 3$, the thing to do is to write explicitly $X_\pm$ and K as 27×27 matrices (this is numerically easy, once we know how to write these generators in $M(3, \mathbb{C}) \oplus (M_{2|1}(\Lambda^2))_0$; this was done in [22] and recalled in [13]) and solve the Eqs. (21), $\|h\|^\dagger G = G\|h^*\|$, for the coefficients G_{ij}, where $h = X_\pm$ and K. One can then check that this set of equations indeed lead to a solution depending on 27 parameters.

Because of this pretty big number of free parameters, the signature can be rather arbitrary. This is a slightly disappointing result since we are looking for some kind of constraint(s) that more or less fix the Hermitian form. It would also be nice if the structure of this scalar product could somehow reflect the algebraic structure of $\mathcal{H}$ itself. As we shall see later, this goal will be achieved by the choice of a particular scalar product that we call the “Hermitian Killing form”. Yet another interesting scalar product on the regular representation can be defined by using the existence of left (or right) invariant integrals (see Section 4.2).

3.5. *Invariant scalar products for $\mathcal{H}$ with a twisted star*

3.5.1. *Invariant scalar products on the indecomposable representations of $\mathcal{H}$*

As it was done for the true Hopf star in Appendix E of [13], we here show the most general metric on the vector space of each of the indecomposable representations of $\mathcal{H}$ using the twisted stars (16). Since we have two possible choices, the $\pm$ signs below correspond, respectively, to the $\pm$ possibilities defined in (15)–(17). We restrict the inner product to be a quantum group invariant one, as defined in Section 3.1. On each representation space we use the basis obtained from appropriate restrictions of the natural basis (“elementary basis”) associated with the regular representation of $\mathcal{H}$ as given in Appendix A. For each indecomposable representation we give an explicit expression of the most general covariant metric in this particular base and we calculate its signature.

- 3_{irr}: up to a real global normalization the metric is

$$G = Diag(1, \mp 1, 1) \quad \text{with signature} \quad \sigma = (+ + \mp).$$

- 6_{odd}: now we get the metric ($\beta \in \mathbb{R}$)

$$G = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & \pm 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & \beta & 1 \\ 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

  A change of basis tells us that $G \sim Diag(1, 1, \pm 1, \pm 1, \lambda_+, \lambda_-)$, with $\lambda_+ > 0$, $\lambda_- < 0$. Thus, the signature is

$$\sigma = (+ + + \pm \pm -).$$

- 5_{odd}: if $\alpha, \beta \in \mathbb{R}$ and $g \in \mathbb{C}$, we may write the metric as

$$G = \begin{pmatrix} \alpha & g & 0 & 0 & 0 \\ \bar{g} & \beta & 0 & 0 & 0 \\ 0 & 0 & \pm\alpha & \pm g & 0 \\ 0 & 0 & \pm\bar{g} & \pm\beta & 0 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

  Given that $G \sim Diag(\lambda_1, \lambda_2, \pm\lambda_1, \pm\lambda_2, 0)$, with arbitrary $\lambda_i \in \mathbb{R}$, its signature may be anything between $\sigma = (+ + + + 0)$ and $\sigma = (- - - - 0)$.

- $3_{\rm odd}$: here

$$G = Diag(1, \pm 1, 0) \quad \text{and} \;\; \sigma = (+ \pm 0).$$

- $6_{\rm eve}$: up to a normalization the metric can be written as ($\beta \in \mathbb{R}$)

$$G = \begin{pmatrix} \beta & 1 & 0 & 0 & 0 & 0 \\ 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & \pm\beta & \pm 1 & 0 & 0 \\ 0 & 0 & \pm 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \mp 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 \end{pmatrix}.$$

The signature is clearly

$$\sigma = (+ + \mp - - -).$$

- $4_{\rm eve}$: in this case the result coincides with the one obtained using the normal Hopf star, as we get the metric ($\alpha, \beta \in \mathbb{R}$, $g \in \mathbb{C}$)

$$G = \begin{pmatrix} 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & \alpha & g \\ 0 & 0 & \bar{g} & \beta \end{pmatrix}.$$

The non-null block in G is an arbitrary Hermitian matrix, therefore the signature is not fixed.

- $3_{\rm eve}$: as in the untwisted case, here we find simply

$$G = Diag(0, 0, 1).$$

- $2_{\rm eve}$: this irreducible representation has the metric

$$G = Diag(1, \pm 1).$$

Notice that a positive definite form is obtained for the twisted Hopf star of $SU(2)$ type.

3.5.2. Invariant scalar product on $\mathcal{M}$

A priori, one could think that the discussion goes along the lines of Section 3.4.2 and that nothing much should be changed. This is almost so, in the sense that invariance implies that the only *possibly non-zero* scalar product of type $(\mathbb{1}, z)$ is $(\mathbb{1}, x^2y^2)$. However, we shall show that this quantity vanishes as well (the proof uses the left action of $\mathcal{H}$ on $\mathcal{M}$ as discussed in Table 1, Section 4.4 of [13]). Indeed

$$\begin{aligned}(\mathbb{1}, x^2y^2) &= (x^2, y^2) = q^{-1}(x^2, Ky^2) = q^{-1}(K^* x^2, y^2) = q^{-1}(K^{-1}x^2, y^2) \\ &= q^{-1}(qx^2, y^2) = q^2q^{-1}(x^2, y^2) = q(\mathbb{1}, x^2y^2)\end{aligned}$$

Hence $(\mathbb{1}, x^2y^2) = 0$, and we see that the bilinear form obtained on $\mathcal{M}$ is totally degenerate. This result contrasts drastically with the one obtained in the untwisted Hopf star case.

3.5.3. Invariant scalar products on the regular representation of $\mathcal{H}$

We refer to Section 3.4.3 for the general discussion and to the next section for a study of very specific scalar products on this representation space.

4. Scalar products on the left regular representation of a Hopf algebra

4.1. The Hermitianized Killing form

As is recalled in Appendix B, in the case of Hopf algebras there is still a notion of a Killing form, which generalizes this particular bilinear form found in the case of Lie groups and algebras. Moreover, it is also invariant under an adequate generalization of the adjoint action of a group on itself, now a left action of a Hopf algebra on itself.

This Killing form $(.,.)_u$ is neither symmetric nor Hermitian (actually we did not use any star in its definition), but, given an arbitrary star operation on H, we now define a sesquilinear form on $H \times H$ by [13]

$$(X, Y) \doteq (X^*, Y)_u = \mathrm{Tr}_q(X^* Y), \quad X, Y \in H. \tag{22}$$

This new form is obviously H-invariant — in the sense of (20) — under the left action of H on itself given by simple multiplication as

$$(XY, Z) = \mathrm{Tr}_q(Y^* X^* Z) = (Y, X^* Z). \tag{23}$$

The "symmetry" property of the Killing form (B.1) gets traduced now in

$$(Y, X) = (X^*, S^2(Y^*)).$$

In addition, if the star operation is a true Hopf one, the invariance of $(\ ,\ )_u$ under the adjoint action (B.2) implies that

$$(ad_{(SZ_1)^*}(X), ad_{Z_2}(Y)) = (X, Y)\,\epsilon(Z). \tag{24}$$

This is so because $[ad_{(SZ)^*}(X)]^* = ad_Z(X^*)$ for a Hopf star. Note that both properties (23) and (24) are invariances of this Killing scalar product in the sense of (18), but with respect to different actions. Actually, we also have for this action a $*$-representation, as it is true that

$$(ad_Z(X), Y) = (X, ad_{Z^*}(Y)).$$

Finally, when the star involved in this definition is a Hopf star, the resulting form is — or can always be chosen to be — Hermitian; we call it the "Hermitianized Killing form" or the "Killing scalar product" (we will see later that this is not the case when one uses a twisted star). Indeed, as we are working with a $*$-representation, $\mathrm{Tr}[h^*] = \overline{\mathrm{Tr}[h]}$, $h \in H$. Therefore,

$$(X, Y) = \mathrm{Tr}(u\, X^* Y) = \overline{\mathrm{Tr}(Y^* X u^*)} = \overline{\mathrm{Tr}(u^* Y^* X)},$$

[13] As for the Killing form, there is always an implicit choice of representation of H, here the left regular one.

and

$$(X, Y) = \overline{(Y, X)} \tag{25}$$

if $u^* = u$. Using the notation of Appendix B, we know that $S^2(h) = uhu^{-1}$ implies, for a Hopf star, $S^2(h) = u^* h (u^*)^{-1}$. Both equations together tell us that $u^{-1}u^*$ is a central element which, being a matrix on a representation space, should be proportional to the identity. Moreover, the proportionality factor must be a phase ($(u^*)^* = u$), and this may always be absorbed in u to have an Hermitian form.

4.1.1. *The Killing scalar product for $\mathcal{H}$ (Hopf star case)*

We have just defined a particular scalar product based on the Killing form on the regular representation of a quantum group H. We analyze here the case of the finite Hopf algebra $\mathcal{H}$, taking $N = 3$, and we choose a Hopf star operation. Then

$$(X, Y) = \mathrm{Tr}_q(X^* Y) = \mathrm{Tr}(K^{-1} X^* Y), \quad X, Y \in \mathcal{H}.$$

In this case, the structure of the corresponding 27×27 Hermitian matrix G in the PBW-basis is not very transparent and we shall not give it explicitly, although its signature can be read off easily. However, the expression of G in what we called the "elementary basis" is quite remarkable. Here are the following:

- Its restriction to the $M(3, \mathbb{C})$ block, with basis ordering

  $$\{E11, E12, E13, E21, E22, E23, E31, E32, E33\},$$

  reads

  $$3 \begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

- Its restriction to the subspace $\{A11, A12, A21, A22\}$ of the $(M_{2|1}(\Lambda^2))_0$ block reads

  $$6 \begin{pmatrix} 0 & 0 & 0 & q \\ 0 & 0 & -q & 0 \\ 0 & -q^{-1} & 0 & 0 \\ q^{-1} & 0 & 0 & 0 \end{pmatrix}.$$

- Its restriction to the subspace $\{A33\}$ of the $(M_{2|1}(\Lambda^2))_0$ block reads

  $$6(1).$$

All other entries vanish. This is in particular so for the scalar products mixing the three aforementioned subspaces. All scalar products also vanish between vectors belonging to the 13-dimensional radical spanned by the generators

$$\{B11, B12, B21, B22, P13, Q13, P23, Q23, P31, Q31, P32, Q32\}.$$

In other words, G is completely degenerated [14] in the direction of the Jacobson radical of $\mathcal{H}$, and it does not mix the different simple components of the semisimple part $\overline{\mathcal{H}} \doteq M(3, \mathbb{C}) \oplus M(2, \mathbb{C}) \oplus \mathbb{C}$. Moreover, we see at once that G restricted to $\overline{\mathcal{H}}$ is diagonal in the (hence orthogonal) basis

$$\{E11 \pm q^{-1}E33, E12 \pm q^{-1}E32, E13 \pm q^{-1}E31,$$
$$E21 \pm q^{-1}E23, E22; A11 \pm q^{-1}A22, A12 \pm q^{-1}A21;\ A33\},$$

where it actually reads

$$G = 3\,Diag(\pm 1, \pm 1, \pm 1, \pm 1, 1;\ \pm 2, \pm 2;\ 1).$$

The signature of the restriction of G to $\overline{\mathcal{H}}$ reads therefore as $(8+, 6-)$, but it is better to write it (with obvious notations) as

$$[4(+1, -1) \oplus (+1)] \oplus [(+1, -1) \oplus (+1, -1)] \oplus (+1).$$

4.1.2. Incompatibility between a Killing scalar product and a twisted Hopf star

Here we can follow a discussion along the same lines of the last part of Section 4.1, but now starting from $S^2(h) = uhu^{-1}$, it is easy to deduce that $S^2(h) = (u^*)^{-1}hu^*$. Both formulas together imply that uu^* is a central element, and this means that we will have $u^* = cu^{-1} \neq u$ ($c \in H$ central).

Therefore, we cannot expect to have an Hermitian Killing scalar product if the star is a twisted one. Having a true (Hermitian) scalar product is incompatible with the invariance of the Killing form.

4.2. Scalar products related to invariant integrals

We first gather general facts and definitions about left- and right-invariant integrals on a Hopf algebra. We then use these concepts — together with a star operation — to define a particular Hermitian scalar product on finite dimensional Hopf algebras. All these notions are illustrated with our favorite example $\mathcal{H}$.

4.2.1. Integrals

A left-invariant integral on a Hopf algebra H over $\mathbb{C}$ is a linear map $\int_{\mathrm{L}} : H \mapsto \mathbb{C}$ such that

$$\left(id \otimes \int_{\mathrm{L}}\right) \circ \Delta = \mathbb{1}_H \int_{\mathrm{L}},$$

[14] The fact that the trace of the adjoint map vanishes on the radical, a result slightly weaker than the one reported here, was separately observed by Kastler [16].

where $\mathbb{1}_H$ is the unit of H and *id* the identity map in H. Therefore, for any $h \in H$ we have (as always $\Delta h = h_1 \otimes h_2$)

$$h_1 \int_{\mathrm{L}} h_2 = \mathbb{1}_H \int_{\mathrm{L}} h. \tag{26}$$

A right-invariant integral $\int_{\mathrm{R}}$ is defined in the obvious similar way.

Since $\int_{\mathrm{L}}$ (or $\int_{\mathrm{R}}$) is a linear object, it can be identified with an element λ_{L} (resp. λ_{R}) of the dual F of H. Such an element will therefore satisfy

$$f\lambda_{\mathrm{L}} = \epsilon(f)\lambda_{\mathrm{L}}$$

(or $\lambda_{\mathrm{R}} f = \epsilon(f)\lambda_{\mathrm{R}}$) for any $f \in F$.

Like for groups, a Hopf algebra H is called *unimodular* if one can find left and right integrals which coincide ($\int \doteq \int_{\mathrm{L}} = \int_{\mathrm{R}}$). Furthermore, such an integral is called a Haar measure when it is normalizable and normalized, i.e., $\int(\mathbb{1}_H) = 1$ (in particular $\int$ should not vanish on the unit!).

We now go back to the example where H is a reduced quantum enveloping algebra of type $SL_q(2, \mathbb{C})$ at a root of unity, $\mathcal{H}$. It is easy to see that here the left and right integrals are respectively given (up to an overall constant) by

$$\int_{\mathrm{L}} = (X_+^{N-1} X_-^{N-1} K)^\star,$$

and

$$\int_{\mathrm{R}} = (X_+^{N-1} X_-^{N-1} K^{-1})^\star.$$

Here a particular vector space basis (PBW) $\{X_+^a X_-^b K^c\}$ is chosen in $\mathcal{H}$ and $\{(X_+^a X_-^b K^c)^\star\}$ denotes its dual basis. In terms of elements of $\mathcal{F}$, the same left- and right-invariant integrals on $\mathcal{H}$ read

$$\lambda_{\mathrm{L}} = (1 + a + \cdots + a^{N-1}) b^{N-1} c^{N-1}, \quad \lambda_{\mathrm{R}} = b^{N-1} c^{N-1} (1 + a + \ldots + a^{N-1}).$$

These two integrals are not proportional and cannot be made equal; $\mathcal{H}$ is therefore not unimodular and no Haar measure can be defined. The dual $\mathcal{F}$ of $\mathcal{H}$ turns out to be unimodular (see [23]), but the corresponding integral is not a Haar measure because it is not normalizable as it vanishes on the unit.

Further restricting now our class of examples to the case $N = 3$, it is interesting to decompose the elements $X_+^2 X_-^2 K$ and $X_+^2 X_-^2 K^{-1}$ on the elementary basis defined in Appendix A. They read, respectively, as

$$\begin{pmatrix} \begin{pmatrix} q^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -q\theta^1\theta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta^1\theta^2 \end{pmatrix} \end{pmatrix},$$

and

$$\begin{pmatrix} \begin{pmatrix} q & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix} & 0 \\ 0 & \begin{pmatrix} -q^2\theta^1\theta^2 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \theta^1\theta^2 \end{pmatrix} \end{pmatrix}.$$

On the other hand, using the PBW basis, the invariant integral [15] on $\mathcal{F}$ can be expressed by duality as the element $\sigma \doteq X_+^2 X_-^2(1 + K + K^2) \in \mathcal{H}$.

4.2.2. Scalar product on the left regular representation

Using both a star operation (any) and an integral on H, we now define a kind of Hopf algebra analog of the familiar scalar product used to discuss square integrable functions in usual complex analysis. We take

$$(X, Y)_{\mathrm{L,R}} \doteq \int_{\mathrm{L,R}} X^* Y, \tag{27}$$

which is then automatically sesquilinear and invariant. In fact, by construction this scalar product satisfies the $*$-representation condition as

$$(ZX, Y) = (X, Z^* Y).$$

Here, H acts on itself by left-multiplication, and the invariance is independent of the star chosen (twisted or not).

Other properties of this scalar product will of course depend upon the kind of star used in its definition.

The Hopf star case

- To have hermiticity of our scalar product we need only to check that

$$\int_{\mathrm{L,R}} X^* = \overline{\int_{\mathrm{L,R}} X},$$

as $(Y, X) = \int (X^*Y)^*$ and $\overline{\int X^*Y} = \overline{(X, Y)}$. It is easy to see that the above property is compatible with the left-invariance of this integral (contrarily to what will happen in the twisted star case). Therefore, one needs to check this explicitly for each case, knowing that a left (or right) invariant integral on a Hopf algebra is unique — if it exists — up to a scalar multiple. We checked explicitly this property for the case of $H = \mathcal{H}$.
- From the invariance property of $\int_{\mathrm{L}}$, one trivially gets

$$\mathbb{1}_H(X, Y) = X_1^* Y_1 (X_2, Y_2).$$

But this may also be interpreted — as happens with the integral — as an invariance with

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[15] Notice that on the group $\mathbb{Z}_3 = \{\mathbb{1}, K, K^2\}$, the integral is given by $\Sigma = 1 + K + K^2$.

respect to the right action of F:

$$(X, Y) \triangleleft f = \epsilon(f)(X, Y) = (X \triangleleft (Sf_1)^*,\ Y \triangleleft f_2).$$

This expression is the analog of (19) for a right action.

Recall that this is an extra invariance of the scalar product, as by construction it is invariant under the left action of H itself.

- In our example of $\mathcal{H}$, with $N = 3$, this Hermitian form expressed in terms of the "elementary basis" defined in Appendix A gives a 27×27 Hermitian matrix G_{ij} that we describe now. Its restriction to the nine-dimensional subspace spanned by

$$\{E_{11}, E_{12}, E_{13}, E_{21}, E_{22}, E_{23}, E_{31}, E_{32}, E_{33}\}$$

reads

$$\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & -q^{-1} & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & q^{-1} & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & -1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & -q & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

Its restriction to the $2(4) + 2(1) = 10$-dimensional subspace spanned by

$$\{A_{11}, B_{11}, A_{12}, B_{12}, A_{21}, B_{21}, A_{22}, B_{22}, A_{33}, B_{33}\}$$

reads

$$\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & 0 & 0 & 0 & -q & -q & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & -q & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & q & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & q & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^{-1} & -q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^{-1} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & -1 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \end{pmatrix}.$$

Finally, its restriction to the eight-dimensional subspace spanned by

$$\{P_{13}, Q_{13}, P_{23}, Q_{23}, P_{31}, Q_{31}, P_{32}, Q_{32}\}$$

reads

$$\frac{1}{3}\begin{pmatrix} 0 & 0 & 0 & q & 0 & 0 & 0 & 0 \\ 0 & 0 & -q & 0 & 0 & 0 & 0 & 0 \\ 0 & -q^2 & 0 & 0 & 0 & 0 & 0 & 0 \\ q^2 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 & 0 & -1 & 0 \\ 0 & 0 & 0 & 0 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 \end{pmatrix}.$$

All the other scalar products vanish.

First of all, we may notice at once that this Hermitian form is not degenerate (this sharply contrasts with the Hermitianized Killing form which is degenerate along the radical, as we saw previously). Here, the signature is $(14+, 13-)$. The 27 eigenvalues themselves read

$$\tfrac{1}{3}\{(1)_9, (-1)_8, (\beta)_2, (-\beta^{-1})_2, (-\beta)_3, (\beta^{-1})_3\},$$

where $\beta = \frac{1}{2}(1+\sqrt{5})$ is the golden number. It is interesting to notice that, although non-degenerate, the restriction of this form to the $9+4+1 = 14$-dimensional semisimple part of $\mathcal{H}$ is positive definite (this part, isomorphic with the matrix algebra $M(3,\mathbb{C}) \oplus M(2,\mathbb{C}) \oplus \mathbb{C}$, as recalled in Appendix A, is spanned by E_{ij} and A_{kl}).

The twisted star case

- It is in general not Hermitian. In fact, if we now write down (26) for h^* and conjugate that equation, we get

$$h_2 \overline{\int_{\mathrm{L}} h_1^*} = \mathbb{1} \overline{\int_{\mathrm{L}} h^*}.$$

If we assume that $\int_{\mathrm{L}} h^* = \overline{\int_{\mathrm{L}} h}$, the above equation would tell us that $\int_{\mathrm{L}}$ should also satisfy the right-invariance condition, which will not be generally true. For instance, in the case of $\mathcal{H}$ we know that a biinvariant integral does not exist. To obtain an Hermitian scalar product we could then add both integrals, $(X, Y) \doteq (\int_{\mathrm{L}} + \int_{\mathrm{R}}) X^* Y$, but this one would not have any extra invariance property.

- From the invariance property of $\int_{\mathrm{L}}$ results

$$\mathbb{1}_H(X, Y) = X_2^* Y_1 (X_1, Y_2),$$

which shows a left–right mixed behavior.

- The example of $\mathcal{H}$, with $N = 3$, is not particularly enlighting since the obtained complex bilinear form is not Hermitian but symmetric. A numerical study of this 27×27 matrix in the elementary basis defined in Appendix A shows that it is not degenerate and that it is “almost” diagonal, in the sense that the only non-diagonal G_{ij} entries are $G(A_{11}, B_{11})$,

$G(A_{12}, B_{12})$, $G(A_{21}, B_{21})$, $G(A_{22}, B_{22})$ and $G(A_{33}, B_{33})$ together with the corresponding symmetric coefficients. We however stress again the fact that, using the twisted star, the scalar product is not Hermitian.

To conclude, the twisted Hopf star case is rather bad in this sense.

5. Discussion

As it was mentioned in Section 1, the parameter q that appears in many integrable and conformal models is often a primitive root of unity, and such values are generally incompatible with the choice of a compact real form on the quantum group (like $SU_q(2)$, for instance). For this reason the stars on "compact" quantum groups that one may define in the context of spin chains, for example, are twisted. The discussion is however a bit subtle and we want to make the following comments.

In the case of a spin chain of type *XXZ*, for instance (see [14], for example), one may start with the *usual* rotation group in three dimensions — or with its double cover $SU(2)$ — acting at each point of the chain. Another ingredient is given by the choice of some (unitary) representation of this group, for instance the fundamental ($s = 1/2$). The Hilbert space of the model is obtained as the nth tensor product of this representation. The Hamiltonian of the model is given by a sum of interaction terms indexed by a discrete label, each term being itself built in terms of (Hermitian) Pauli matrices. This Hamiltonian is *not*, in general, invariant with respect to the rotation group since the physical system is clearly not rotationally invariant. However, in some cases, one notices that the same total Hamiltonian commutes with the generators of a (complex) quantum group, for instance $U_q(sl(2, \mathbb{C}))$. We should stress the fact that generators of $SU(2)$ act on the Hilbert space in a way that is "local" (generators rotate the states independently at each point of the chain), whereas $U_q(sl(2, \mathbb{C}))$ acts in a non-local way (this point of view was emphasized for instance in [24]). Notice that hermiticity of the Hamiltonian — a Jones projector — is clearly a required constraint, however, this property does not take place in a representation space for the quantum group but in its commutant.

Both $SU(2)$ and $U_q(sl(2, \mathbb{C}))$ enter the discussion of the model and both have two-dimensional representations, but the two related concepts should not be confused. For physical reasons, it is clear that the scalar product used on the Hilbert space of the model should not contain vectors of negative norm; for this reason it should be a bonafide positive definite scalar product. The same Hilbert space could also be built in terms of tensor products of the fundamental representation of the quantum group $U_q(sl(2, \mathbb{C}))$, for q a root of unity; indeed, two vector spaces over $\mathbb{C}$ of the same dimension are clearly isomorphic as vector spaces. Nevertheless, in the usual construction the Hilbert space of the model acquires its Hilbert structure from the scalar product chosen on representations of $SU(2)$, not from the one chosen on the representations of the quantum group. Actually, the authors of the present paper do not see why such a choice should be performed at all; they cannot exclude however that it may turn out to be useful. What is in any case clear is that if one wants to choose a scalar product on the fundamental representation of $U_q(sl(2, \mathbb{C}))$ such that it will induce the same (already given) positive scalar product on the Hilbert space of the model, one has to suppose that the quantum group is endowed with a star operation which is a twisted Hopf star of $SU(2)$ type.

We should mention the papers [5,6], where a general study of quantum symmetries in quantum theory is done, and where the *choice* of twisted star operations is clearly made right at the beginning. This was actually nothing else than a choice (related to a way of defining a covariant adjoint for field operators), and it was subsequently discovered [16] that this choice was not unique and that it would have been also perfectly possible to define adjoints for field operators after having decided to use a "true" Hopf star operation.

In conformal theories, primary fields are associated with vectors of highest weight in a representation of some affine algebra, and it was observed long ago that the fusion table of such primary fields is identical to the Clebsh Gordan table describing the tensor products of irreducible representations of some quantum group — the same quantum group also appears, via its $6j$-symbols, in the equations describing the duality properties of the conformal blocks. At this point, one should stress that the representations of the quantum group that appear in the associated fusion table are not to be confused with the representations of the affine algebra. The two structures, although related (in a way that is apparently not well understood yet, see [25]), are quite distinct and the discussion involving the nature of the scalar product to be used in a given representation space for the affine or Virasoro generators should not be confused with the analysis of the scalar product(s) that one can define on the modules of the emerging quantum group.

When the parameter q is a root of unity, the representation theory is quite subtle since indecomposable (but not irreducible) representations of the quantum group appear. Actually, to obtain a physically meaningful state space one has to choose a so-called "truncated tensor product", by selecting only those representations for which the q-trace vanishes (one can also use the formalism of quasiHopf algebras, see [6]). It is a fact that discussions involving quantum groups in conformal field theories usually consider *infinite dimensional* Hopf algebras (like $U_q(sl(2,\mathbb{C}))$), which are not "good" quantum groups when q is a root of unity since they are not quasitriangular in the usual sense. At the contrary, the *finite dimensional* Hopf algebras that one can obtain from those ones through division by an (infinite dimensional) Hopf ideal are not semisimple but they are quasitriangular: they possess (finite dimensional) R-matrices. The category of representations of these Hopf algebras is not a modular category (tensor products of irreducible representations are not necessarily equivalent to direct sums of irreducibles), but it is again possible to define truncated scalar products in a very natural way. We conjecture that discussions involving simultaneously rational conformal field theories and quantum groups should be done in terms of such finite dimensional Hopf quotients of the usual quantum enveloping algebras at roots of unity. A general study of these topics stays outside the scope of the present paper but we hope that our contribution concerning stars (twisted or not) and scalar products, together with selected examples involving finite dimensional Hopf algebra quotients of $U_q(sl(2,\mathbb{C}))$ will be useful in this respect.

[16] Unpublished addendum by the same authors. We thank G. Mack for this information.

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Appendix A. Structure of the reduced Hopf algebra $\mathcal{H}$

When q is a root of unity ($q^N = 1$), the quantized enveloping algebra $U_q(sl(2,\mathbb{C}))$ possesses interesting quotients that are finite dimensional Hopf algebras. The structure of the left regular representation of such an algebra was investigated in [26] and the pairing with its dual in [27]. We call $\mathcal{H}$ the Hopf algebra quotient of $U_q(sl(2,\mathbb{C}))$ defined by the relations

$$K^N = \mathbb{1}, \qquad X_\pm^N = 0,$$

and $\mathcal{F}$ its dual. The generators K, $X_\pm$ are chosen to obey the following commutation and cocommutation relations:

Product:

$$KX_\pm = q^{\pm 2} X_\pm K, \qquad [X_+, X_-] = \frac{1}{(q-q^{-1})}(K - K^{-1}),$$
$$K^N = \mathbb{1}, \qquad X_+^N = X_-^N = 0 \tag{A.1}$$

Coproduct:

$$\Delta X_+ = X_+ \otimes \mathbb{1} + K \otimes X_+, \qquad \Delta X_- = X_- \otimes K^{-1} + \mathbb{1} \otimes X_-,$$
$$\Delta K = K \otimes K, \qquad \Delta K^{-1} = K^{-1} \otimes K^{-1} \tag{A.2}$$

It was shown [17] in [26] that the non-semisimple algebra $\mathcal{H}$ is isomorphic with the direct sum of a complex matrix algebra and of several copies of suitably defined matrix algebras with coefficients in the ring $Gr(2)$ of Grassmann numbers with two generators. The explicit structure of those algebras (for any N), including the expression of generators themselves, was obtained by Ogievetsky [28]. Using these results, the representation theory of $\mathcal{H}$ for the case $N = 3$ was presented in [22].

When $q^N = 1$ with N odd, [18] we have an isomorphism between the N^3-dimensional algebra $\mathcal{H}$ and the direct sum

$$\mathcal{H} = M_N \oplus (M_{N-1|1}(\Lambda^2))_0 \oplus (M_{N-2|2}(\Lambda^2))_0 \oplus \cdots \oplus (M_{(N+1)/2|(N-1)/2}(\Lambda^2))_0, \tag{A.3}$$

[17] Alekseev et al. [26] actually consider a Hopf algebra quotient defined by $K^{2N} = \mathbb{1}$, $X_\pm^N = 0$, so their algebra is, in a sense, twice bigger than ours (see Appendix C).

[18] When N is even with $N' = N/2$ odd, $K^{N'}$, $X_\pm^{N'}$ are central and one may take the quotient by $K^{N'} = \mathbb{1}$, $X_\pm^{N'} = 0$; the algebra so obtained is isomorphic with $\mathcal{H}$. When $N' = N/2$ is even the structure is quite different, and we do not study it here (see [28]).

where
1. M_N is an $N \times N$ complex matrix,
2. an element of the $M_{N-2|2}$ block (for instance) is of the kind:

$$\begin{pmatrix} \bullet & \bullet & \cdots & \bullet & \circ & \circ \\ \bullet & \bullet & \cdots & \bullet & \circ & \circ \\ \vdots & \vdots & & \vdots & \vdots & \vdots \\ \bullet & \bullet & \cdots & \bullet & \circ & \circ \\ \circ & \circ & \cdots & \circ & \bullet & \bullet \\ \circ & \circ & \cdots & \circ & \bullet & \bullet \end{pmatrix}, \tag{A.4}$$

where we have introduced the following notation: $\bullet$ is an even element of the ring $Gr(2)$ of Grassmann numbers with two generators, [19] i.e., of the kind:

$$\bullet = \alpha + \beta\theta_1\theta_2, \quad \alpha, \beta \in \mathbb{C},$$

$\circ$ is an odd element of the ring $Gr(2)$, i.e., of the kind:

$$\circ = \gamma\theta_1 + \delta\theta_2, \quad \gamma, \delta \in \mathbb{C},$$

etc.

Notice that $\mathcal{H}$ is not a semisimple algebra: its Jacobson radical $\mathcal{J}$ is obtained by selecting in Eq. (A.3) the matrices with elements proportional to Grassmann variables. The quotient $\mathcal{H}/\mathcal{J}$ is then semisimple... but no longer Hopf!

Projective indecomposable modules (PIMs, also called principal modules) for $\mathcal{H}$ are directly given by the columns of the previous matrices.

3. From the M_N block, one obtains N equivalent irreducible representations of dimension N that we shall denote by N_{irr}. These representations have vanishing q-dimension.
4. From the $M_{N-p|p}$ block ($p < N - p$), one obtains
   4.1. $(N - p)$ equivalent indecomposable projective modules of dimension $2N$ that we shall denote by P_{N-p} with elements of the kind

$$(\underbrace{\bullet\,\bullet \cdots \bullet}_{N-p}\underbrace{\circ\,\circ \cdots \circ}_{p}). \tag{A.5}$$

   4.2. p equivalent indecomposable projective modules (also of dimension $2N$) that we shall denote by P_p with elements of the kind

$$(\underbrace{\circ\,\circ \cdots \circ}_{N-p}\underbrace{\bullet\,\bullet \cdots \bullet}_{p}). \tag{A.6}$$

These PIMs have also q-dimension equal to zero. To each PIM P_s is associated an irreducible representation of dimension s, obtained by quotienting P_s by its own radical. These irreps have non-vanishing q-dimension, and are in one-to-one correspondence with the so-called type II irreducible representations of $U_q(sl(2, \mathbb{C}))$.

[19] Remember that $\theta_1^2 = \theta_2^2 = 0$ and $\theta_1\theta_2 = -\theta_2\theta_1$.

Other submodules can be found by restricting the range of parameters appearing in the columns defining the PIMs and imposing stability under multiplication by elements of $\mathcal{H}$. In this way one can determine for each PIM the lattice of its submodules. For each PIM of dimension $2N$, one finds totally ordered sublattices with exactly three non-trivial terms: the radical (here, it is the biggest non-trivial submodule of a given PIM), the socle (here it is the smallest non-trivial submodule), and one “intermediate” submodule of dimension exactly equal to N. However, the definition of this last submodule (up to an equivalence) depends on the choice of an arbitrary complex parameter λ, so that we have a chain of inclusions for every such parameter. The collection of all these sublattices fully determines the lattice structure of submodules of a given principal module.

We are interested in this paper in Hopf stars (twisted or not) and invariant scalar products for representation spaces of $\mathcal{H}$. To ease the presentation of the results, it is better to limit ourselves to the case $N = 3$ but the overall picture should be clear. From now on, we take $N = 3$.

In the case $q^3 = 1$, $\mathcal{H}$ is a 27-dimensional Hopf algebra isomorphic with $M(3, \mathbb{C}) \oplus (M_{2|1}(\Lambda^2))_0$. Explicitly,

$$\mathcal{H} = \left\{ \begin{pmatrix} e_{11} & e_{12} & e_{13} \\ e_{21} & e_{22} & e_{23} \\ e_{31} & e_{32} & e_{33} \end{pmatrix} \oplus \begin{pmatrix} \alpha_{11} + \beta_{11}\theta_1\theta_2 & \alpha_{12} + \beta_{12}\theta_1\theta_2 & \gamma_{13}\theta_1 + \delta_{13}\theta_2 \\ \alpha_{21} + \beta_{21}\theta_1\theta_2 & \alpha_{22} + \beta_{22}\theta_1\theta_2 & \gamma_{23}\theta_1 + \delta_{23}\theta_2 \\ \gamma_{31}\theta_1 + \delta_{31}\theta_2 & \gamma_{32}\theta_1 + \delta_{32}\theta_2 & \alpha_{33} + \beta_{33}\theta_1\theta_2 \end{pmatrix} \right\}. \tag{A.7}$$

All entries besides the θ’s are complex numbers (the above $\oplus$ sign is a direct sum sign: these matrices are 6×6 matrices written as a direct sum of two blocks of size 3×3).

The semisimple part $\bar{\mathcal{H}}$, given by the direct sum of its block-diagonal θ-independent parts, is equal to the $9 + 4 + 1 = 14$-dimensional algebra $\bar{\mathcal{H}} = M_3(\mathbb{C}) \oplus M_2(\mathbb{C}) \oplus \mathbb{C}$. The radical (more precisely the Jacobson radical) J of $\mathcal{H}$ is the left-over piece that contains all the Grassmann entries, and only the Grassmann entries, so $\bar{\mathcal{H}} = \mathcal{H}/J$. The radical has therefore dimension 13.

PIMs are given by the columns of the previous expression. We see that the left regular representation splits into a sum of three equivalent three-dimensional projective indecomposable representations that we call 3_{irr} (they are also irreducible) given by the columns of $M(3, \mathbb{C})$, two equivalent six-dimensional projective indecomposable representations that we call 6_{eve} given by the first two columns of $(M_{2|1}(\Lambda^2))_0$ and one six-dimensional projective indecomposable representation that we call 6_{odd} given by the last column of $(M_{2|1}(\Lambda^2))_0$. The left regular representation can therefore be decomposed as follows:

$$3[3_{\text{irr}}] \oplus 2[6_{\text{eve}}] \oplus 1[6_{\text{odd}}].$$

All these projective indecomposable representations have zero quantum dimension.

Irreducible representations are obtained by taking the quotient of the projective indecomposable ones by their respective radical (killing the Grassmann variables). One obtains in this way the irreducible representation 3_{irr} that we already had, a two-dimensional irreducible 2_{irr} (quotient of 6_{eve}) and a one-dimensional irreducible 1_{irr} (quotient of 6_{odd}).

Notice that $2_{\rm irr}$ and $1_{\rm irr}$ do not have vanishing quantum dimension, whereas as we already mentioned the $3_{\rm irr}$ is special in this respect, since it is also one of the PIMs.

In order to discuss the results it is convenient to select a particular linear basis in $\mathcal{H}$. Actually, three of them turn out to be quite useful. The first one, the "PBW-basis", is given (up to ordering) by the set of monomials $X_+^a X_-^b K^c$.

The second one, that we shall call the "elementary basis", comes from the previous isomorphism with $M(3,\mathbb{C})\oplus(M_{2|1}(\Lambda^2))_0$. We call E_{ij} the elementary matrices corresponding to the $M(3,\mathbb{C})$ block (they correspond to the e_{ij} coefficients of (A.7)). As for the $(M_{2|1}(\Lambda^2))_0$ block, we call A_{ij}, B_{ij}, P_{ij}, Q_{ij} the elementary matrices corresponding to the α_{ij}, β_{ij}, γ_{ij}, δ_{ij} coefficients, respectively. Clearly, this set of elementary matrices is also a basis of $\mathcal{H}$ and it is not too difficult (though it is cumbersome) to express each of its elements in terms of the PBW-basis.

The last useful basis, directly related to the elementary basis, is defined in Section 4.1.1, it has the property of diagonalizing the "Hermitianized" Killing form.

Appendix B. The Killing form on a quantum group

B.1. *The adjoint representation of a quantum group*

If $X\in H$, then the adjoint map $ad_X : H\mapsto H$ is defined by

$$ad_X(Y)\doteq X_1\, Y\, S(X_2).$$

Notice that this definition generalizes both the notion of adjoint representation for groups (where $\Delta g = g\otimes g$ and $S(g)=g^{-1}$, g being a group element) and for Lie algebras (where $\Delta X = X\otimes\mathbb{1}+\mathbb{1}\otimes X$ and $S(X)=-X$, X being a Lie algebra element).

The representation ad is a left action. It is indeed easy to show that

$$ad_{XY}(Z) = ad_X(ad_Y(Z)).$$

Actually, it is also possible to define "another" adjoint representation by replacing the previous definition by $S(X_1)YX_2$; this is not a left action but a right one (so it can be called the "right"-adjoint action).

One could be tempted to consider the right action $S^{-1}(X_1)YX_2$ or the left action $X_1YS^{-1}(X_2)$ but these actions are not compatible with the algebra structure (indeed acting on the unit with some element X would not give $\epsilon(X)\,\mathbb{1}$). Moreover, it is not very useful to consider the left and right actions $X_2YS^{-1}(X_1)$ and $S^{-1}(X_2)YX_1$ since, although compatible with the algebra structure, they are essentially equivalent with the previously given definitions for the left and right adjoint actions. In fact the antipode intertwines both maps.

In the sequel, we shall only use the first definition of the adjoint action, we should therefore remember that it is a left action.

The adjoint action is compatible with the algebra structure. One indeed shows that

$$ad_X(YZ) = ad_{X_1}(Y)\, ad_{X_2}(Z).$$

Notice that the two given properties allow one to easily compute the explicit expression for the adjoint representation once it is known on the generators.

Case of $\mathcal{H}$. In this case, one obtains easily the adjoint action on the generators:

$$ad_K(K) = K, \qquad ad_{X_-}(K) = (1-q^{-2})X_-K^2, \qquad ad_K(X_-) = q^{-2}X_-,$$
$$ad_{X_-}(X_-) = 0, \qquad ad_K(X_+) = q^2X_+, \qquad ad_{X_-}(X_+) = \frac{\mathbb{1}-K^2}{q-q^{-1}},$$
$$ad_{X_+}(K) = (1-q^2)X_+K, \qquad ad_{X_+}(X_+) = (1-q^2)X_+^2$$
$$ad_{X_+}(X_-) = (1-q^{-2})X_+X_- + \frac{K-K^{-1}}{q^3-q}.$$

B.2. The quantum trace

If H is a quasitriangular Hopf algebra, with an universal R-matrix $\mathcal{R}$, there exists in it a special element

$$u_0 \doteq m[(S\otimes id)\mathcal{R}_{21}].$$

Such an element is invertible and allows to write explicitly S^2 as an inner automorphism (see [2] for a proof and a more general discussion):

$$S^2(h) = u_0 h u_0^{-1}, \quad \forall h \in H.$$

On the other hand, given ρ a representation of H on a space V, the quantum trace is the map defined by the following chain of isomorphisms, all of them commuting with the H-action:

$$End(V) \to V\otimes V^{\star} \to V^{\star\star}\otimes V^{\star} \to \mathbb{C}.$$

Remember that given a representation on V, one obtains naturally a representation on its dual space $V^{\star}$, by making use of the antipode ($h \triangleright v^{\star}$ is such that $\langle h\triangleright v^{\star}, w\rangle = \langle v^{\star}\, S(h)\triangleright w\rangle\ \forall w \in V$). The non-canonical isomorphism $V \simeq V^{\star\star}$ given by $v \to \rho(u_0)v$ is needed in order to make the chain commute with the action of the quantum group. Therefore, the resulting expression for the quantum trace in terms of the ordinary operator trace on V is

$$\mathrm{Tr}_q(X) = \mathrm{Tr}(\rho(u_0)X), \quad X \in End(V).$$

As u_0 has no reason to be group-like, this trace is in general not multiplicative on tensor products of representations of H, but can be made so if H is a ribbon Hopf algebra. In this case there exists an invertible and central element $v \in H$ such that $v^2 = u_0S(u_0)$, $S(v) = v$, and $\Delta v = (\mathcal{R}_{21}\mathcal{R}_{12})^{-1}(v\otimes v)$. Now u_0 may be replaced in Tr_q by $u \doteq v^{-1}u_0$, which is group-like. It is still true that $S^2(h) = uhu^{-1}$, because v is central.

In the case of $H = \mathcal{H}$ we find $u = K^{-1}$ (and $v = \mathbb{1}$).

B.3. The Killing form

Let X, Y denote two matrices (with $\mathbb{C}$-number entries!) representing elements X and Y of a Hopf algebra H in some representation (we keep the same notation, here, for elements of the Hopf algebra and their matrix representatives). The Killing form in this representation is defined by

$$(X, Y)_u \doteq \mathrm{Tr}_q(XY) = \mathrm{Tr}(uXY).$$

The terminology "Killing form" usually refers to a particular bilinear form on a Lie algebra and its representations. Extension of this notion to the enveloping associative algebra is usually not considered. In the present case, we are therefore using a slightly generalized terminology (like in [29]). Notice that in our examples the Hopf algebra H is finite dimensional, so we can even discuss the structure of this Killing form in the regular representation.

Symmetry of the Killing form. As $S^2(X) = uXu^{-1}$, then

$$(X, S^2(Y))_u = (X, uYu^{-1})_u = \mathrm{Tr}(uXuYu^{-1}) = \mathrm{Tr}(XuY) = \mathrm{Tr}(uYX).$$

Therefore,

$$(Y, X)_u = (X, S^2(Y))_u. \tag{B.1}$$

This reduces to the usual symmetry when S^2 is the identity, which is in particular the case for a group.

Invariance of the Killing form under the adjoint action. One can show that

$$(ad_{Z_1}(X), ad_{Z_2}(Y))_u = (X, Y)_u\, \epsilon(Z). \tag{B.2}$$

In the classical case of a group or a Lie algebra, this reduces to the usual invariance of the Killing form under the adjoint action.

To prove this property, one needs the following lemma:

$$\mathrm{Tr}(u\, ad_X(Y)) = \mathrm{Tr}(uY)\, \epsilon(X).$$

Indeed,

$$\begin{aligned}\mathrm{Tr}(u\, ad_X(Y)) &= \mathrm{Tr}(uX_1YS(X_2)) = (X_1, YS(X_2))_u = (YS(X_2), S^2(X_1))_u\\ &= \mathrm{Tr}(uYS(X_2)S^2(X_1)) = \mathrm{Tr}(uY)\, \epsilon(X).\end{aligned}$$

Therefore, the left-hand side of (B.2) reads

$$\mathrm{Tr}(u\, ad_{Z_1}(X)\, ad_{Z_2}(Y)) = \mathrm{Tr}(u\, ad_Z(XY)) = \mathrm{Tr}(u\, XY)\, \epsilon(Z) = (X, Y)_u\, \epsilon(Z).$$

Appendix C. The "double" $\tilde{\mathcal{H}}$ of $\mathcal{H}$

We now take $q^N = 1$ (N odd), as before, but consider the finite dimensional quotient $\tilde{\mathcal{H}}$ of the quantum algebra $U_q(sl(2, \mathbb{C}))$ by the Hopf ideal defined by $X_\pm^N = 0$, $K^{2N} = \mathbb{1}$

(rather than $K^N = \mathbb{1}$). Notice that this "double" has nothing to do with what is called the "quantum double" of a Hopf algebra in the literature.

In order to make use of all the results concerning $\mathcal{H}$, take $X_\pm$ and K as the generators of $\mathcal{H}$, as before, and call $\tilde{X}_\pm$ and $\tilde{K}$ the generators of $\tilde{\mathcal{H}}$. Now set

$$\tilde{K} = \sigma_3 \otimes K = Diag(K, -K), \qquad \tilde{X}_+ = \mathbb{1} \otimes X_+ = Diag(X_+, X_+),$$
$$\tilde{X}_- = \sigma_3 \otimes X_- = Diag(X_-, -X_-), \tag{C.1}$$

where σ_i are the Pauli matrices. This provides an explicit realization of $\tilde{\mathcal{H}}$ in terms of $\mathcal{H}$. One sees immediately that $\dim(\tilde{\mathcal{H}}) = 2\dim(\mathcal{H}) = 2N^3$ and obtains also for $N = 3$ an explicit expression for the generators, in terms of Grassmann valued 12×12 matrices, by using the expressions of $X_\pm$, K given in [22] or [13]. By construction, it is clear that $\mathcal{H}$ is a $\mathbb{Z}_2$ quotient of $\tilde{\mathcal{H}}$ — notice that the group generated by powers of $\tilde{K}$ is no longer $\mathbb{Z}_3$, like before, but $\mathbb{Z}_3 \times \mathbb{Z}_2$ and that $\tilde{K}^3$ is a non-trivial central element.

The representation theory of this algebra can then be obtained in a straightforward manner: projective indecomposable representations are still given by the columns of the corresponding isomorphic Grassmann valued matrix algebra; the ones appearing in the upper diagonal 6×6 block of (C.1) are the same 3_{irr}, 6_{odd} and 6_{eve} considered in Appendix A; those appearing in the lower block will be denoted by 3^-_{irr}, 6^-_{odd} and 6^-_{eve}. More generally (for arbitrary N), we see that indecomposable representations of $\tilde{\mathcal{H}}$ are of two kinds: they can be labeled by $\omega = \pm 1$, those for which $\omega = 1$ are *also* representations of $\mathcal{H}$, whereas those for which $\omega = -1$ only appear as representations of $\tilde{\mathcal{H}}$. These two kinds of representations can therefore be distinguished by the eigenvalue of the non-trivial central element $\tilde{K}^3$. Remark that, when $\tilde{\mathcal{H}}$ is (faithfully) realized, as explained above, in terms of 12×12 matrices with Grassmann entries, the restrictions $K|_1$ and $K|_2$ of $\tilde{K}$ to the upper and lower blocks are such that $\tilde{K}|_1^3 = \mathbb{1}_{3\times 3}$, and $\tilde{K}|_2^3 = -\mathbb{1}_{6\times 6}$.

It may be useful to recall that, when q is an odd (N) root of unity, the center of $U_q(sl(2,\mathbb{C}))$ is generated by the Casimir C, $X_\pm^N$ and $K^{\pm N}$. Call c, x, y and $z^{\pm 1}$ the values of these central elements in irreducible representations. There are irreducible representations "of classical type" usually denoted by $\mathrm{Spin}(j, \omega)$, where j is a half-integer spin and $\omega = \pm 1$; in those representations $x = y = 0$ and $z = \omega^N = \pm 1$. There are also irreducible representations "of non-classical type" which can be "periodic" ($xy \neq 0$) or semiperiodic ($xy = 0$ but either $x \neq 0$ or $y \neq 0$); such representations do not appear for finite dimensional Hopf algebras quotients such as $\mathcal{H}$ since both x and y will then automatically vanish. Somehow, considering $\tilde{\mathcal{H}}$ instead of $\mathcal{H}$ has the interest of allowing one to recover also the irreducible representations of $U_q(sl(2,\mathbb{C}))$ with $\omega = -1$ as representations of a quasitriangular *finite* dimensional Hopf algebra.

A general discussion concerning Hopf stars, twisted or not, and scalar products can be done here along the same general lines as before. In particular, notice that when we choose one of the two possible twisted Hopf stars ($X_+^* = \pm X_-$, $X_-^* = \pm X_+$, $K^* = K^{-1}$), the invariant scalar products associated with the family of corresponding star representations ($\omega = \pm 1$) of $\tilde{\mathcal{H}}$ simultaneously exhibit features that in the case of $\mathcal{H}$ were obtained separately for (twisted) stars of type $SU(2)$ or $SU(1,1)$. For example, we know (see Section 3.5.1) that the invariant scalar product on 3_{irr} (i.e., $\omega = +1$) associated with the $SU(2)$ twisted Hopf star is of signature $(+ + -)$, and that the signature is $(+ + +)$ for the twisted star of type $SU(1, 1)$. It happens that the conclusions are just to be reversed when we replace 3_{irr} by $3^-_{\mathrm{irr}} (\omega = -1)$. It may also be of interest to notice that invariant scalar products corresponding to irreducible representations 3^-_{irr} and 2_{eve} (for the twisted $SU(2)$ case), or 3_{irr} and 2^-_{eve} (for the twisted $SU(1, 1)$ case) of this double $\tilde{\mathcal{H}}$ have a positive definite metric.

Regarding invariant scalar products on the left regular representation of $\tilde{\mathcal{H}}$ (as a module-algebra), we have a freedom of 54 real parameters, for the same reasons as those given in Section 3.4.3, but specific scalar products can be defined as in Section 4.

Remark (The simply connected form of $U_q(sl(2, \mathbb{C}))$). *A standard construction at the level of the infinite dimensional universal quantum algebra* (*see for example* [2,3]) $U_q(sl(2, \mathbb{C}))$ *consists in introducing a square root* k for K, *so that* $k^2 = K$; *it is also useful to define generators* $I_\pm$ *for which the coproduct is symmetrical, i.e.,* $\Delta I_\pm = I_\pm \otimes k^{-1} + k \otimes I_\pm$. *This infinite dimensional algebra generated by* $\{k, I_\pm\}$ *is often called the "simply connected form" of* $U_q(sl(2, \mathbb{C}))$ *and denoted* $\check{U}_q(sl(2, \mathbb{C}))$. (*In the literature this object is sometimes called just* $SL_q(2)$!) $U_q(sl(2, \mathbb{C}))$ *is a Hopf subalgebra of* $\check{U}_q(sl(2, \mathbb{C}))$; *the explicit inclusion of the former in the latter can be obtained by taking* $K = k^2$, $X_+ = I_+ k$ *and* $X_- = k^{-1} I_-$. *Since* k^3 *is central, one could then be tempted to build a finite dimensional Hopf quotient of* $\check{U}_q(sl(2, \mathbb{C}))$ *by factoring it by the ideal given by* $I^3_\pm = 0$ *and* $k^3 = \mathbb{1}$. *The point is that one does not get anything essentially new by doing so: the obtained quotient is isomorphic with* $\mathcal{H}$ *itself. Indeed, let us set* $K \doteq k^2$ *at the level of this quotient, then* $K^2 = k^4 = k$ and $K^3 = k^6 = \mathbb{1}$. *Hence the relation between* k *and* K *can be inverted. Moreover, one can check explicitly* (*thanks to the previously given change of variables between* $X_\pm$ *and* $I_\pm$) *that all the algebra and coalgebra relations of this finite dimensional quotient of* $\check{U}_q(sl(2, \mathbb{C}))$ *coincide exactly with those given for* $\mathcal{H}$ *itself.*

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